

2012 Mathematics (1)

This pdf was generated from questions and answers contributed by members of the public to Christopher Lester's tripos/example-sheet solution exchange site <http://cgl20.user.srcf.net/>. Nothing (other than raven authentication) prevents rubbish being uploaded, so this pdf comes with no warranty as to the correctness of the questions or answers contained. Visit the site, vote, and/or supply your own content if you don't like what you see here.

This pdf had url <http://cgl20.user.srcf.net/camcourse/paperpdf/12?withSolutions=1>.

This pdf was created on Thu, 18 Apr 2024 22:23:48 +0000.

Section A

1

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

2

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

3

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

4

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

5

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

6

Express $\frac{4x^2 - x}{(x - 1)^2(x + 2)}$ as partial fractions.

[2]

Solution(s):

From user: lester

2012 Paper 1, 1A MNST, (6).

$$\frac{4x^2 - x}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \quad (*)$$

"x by (x+2)" and let $x \rightarrow -2$. get. $\frac{4 \cdot 4 + 2}{(-3)^2} = C \Rightarrow C = 2$.

$$\text{"x (x-1)" } \Rightarrow \frac{4x^2 - x}{(x-1)(x+2)} = A + \frac{B}{x-1} + \frac{C(x-1)}{x+2}.$$

$$\text{Let } x \rightarrow \infty. \text{ Then } 4 = A + 0 + C \Rightarrow A = 4 - C = 4 - 2 = 2$$

Finally "x (x-1)^2" \Rightarrow

$$\frac{4x^2 - x}{x+2} = A(x-1) + B + \frac{C(x-1)^2}{x+2}$$

$$\text{and let } x \rightarrow 1 \text{ giving } \frac{4-1}{3} = 0 + B + 0 \Rightarrow B = \underline{1}$$

$$\text{So: } (*) = \frac{2}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{x+2}.$$

7

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

8

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

9

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

10

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

Section B

11X

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

12Z

No image has yet been uploaded for this question

Solution(s):

From user: ar857

2012 Paper 1 Q 12Z

12Z

~~$(\frac{1+i}{1-i})^2 = \frac{(1+i)^2}{(1-i)^2} = \frac{1+2i+i^2}{1-2i+i^2} = \frac{2i}{-2i} = -1$~~

a) i) $\left(\frac{2+i}{-1+i}\right)^2 = \left(\frac{2+i}{-1+i} \cdot \frac{-1-i}{-1-i}\right)^2 = \left(\frac{-2-i-2i-i^2}{2}\right)^2 = \left(\frac{-2-3i-i^2}{2}\right)^2 = \left(\frac{-1-3i}{2}\right)^2$
 $= \frac{1}{4} (1-9+6i) = -2 + \frac{3}{2}i$

ii) $(1+i)^{10} = \left(\sqrt{2}\left(\frac{1}{2} + \frac{1}{2}i\right)\right)^{10} = 32(0+i) = \underline{32i}$

iii) $\sin\left(\frac{\pi}{2} + i\ln 2\right) = \sin \frac{\pi}{2} \cosh \ln 2 + \cos \frac{\pi}{2} \sinh \ln 2$
 $= \cosh(\ln 2) = \frac{e^{\ln 2} + e^{-\ln 2}}{2} = \frac{2 + \frac{1}{2}}{2} = \frac{5}{4}$

b) $z^2 = (i-1) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) = \sqrt{2} \cos\left(\frac{3}{4}\pi + 2\pi\right) + i \sin\left(\frac{3}{4}\pi + 2\pi\right)$
 $z = \sqrt[4]{2} \left[\cos\left(\frac{3}{8}\pi + n\pi\right) + i \sin\left(\frac{3}{8}\pi + n\pi\right) \right]$

c) $\sin^5 \phi = \frac{1}{32}(i)^5 \cdot (e^{\phi} - e^{-\phi})^5$
 $= \frac{1}{16} \cdot \frac{1}{2i} (e^{5\phi} - 5e^{4\phi}e^{-\phi} + 10e^{3\phi}e^{-2\phi} - 10e^{2\phi}e^{-3\phi} + 5e^{\phi}e^{-4\phi} - e^{-5\phi})$
 $= \frac{1}{16} (\sin 5\phi - 5\sin 3\phi + 10\sin \phi)$

d) $\cosh z = -1$
 $A + \frac{1}{A} = -1$
 $A^2 + 2A + 1 = 0$
 $A = -1$
 $z = \ln(-1) = \ln(e^{(\pi+2\pi)i})$

2012 Paper I Q 13

13Z

No image has yet been uploaded for this question

Solution(s):

From user: ar857

2012 Paper I Q13

(13) $\frac{dy}{dx} = y^2 + xy = \frac{y^2}{x^2} + \frac{y}{x}$

$z = \ln(-1) = \ln(e^{(\pi+2\pi i)})$

$z = \pi i (1+2n)$

$A^2 + 2A + 1 = 0$

SUBSTITUTION

$y = vx$

$V = \frac{y}{x}$

$\frac{dy}{dx} = \frac{dv}{dx}x + v$

a) $\frac{dv}{dx}x + v = v^2 + v$

$\frac{dv}{v^2} = \frac{1}{x} dx$

$-\frac{1}{v} = \ln x + c$

$-\frac{x}{y} = \ln x + c$

$y = \frac{-x}{\ln x + c}$

b) $\frac{d(x+y)}{dy} = 1 = \frac{d(x)}{dx} = 1 \Rightarrow$ exact differential

$x \frac{dy}{dx} + x + y = 0$

general solution $\frac{1}{2}x^2 + yx = c$

$y = \frac{(c - \frac{1}{2}x)}{x}$

From user: ar857

c)

$\frac{dy}{dx} + ky = a \sin mx$

$\int e^{kx} \frac{d}{dx} (e^{-kx} y) = \int a \sin mx e^{-kx} dx$

$e^{kx} y = \int a \sin mx e^{-kx} dx$

$= a \int \sin mx e^{-kx} dx = \left(\sin mx \frac{1}{k} e^{-kx} - \int \frac{m}{k} e^{-kx} \cos mx \right) / a$

$= a I_{mk} = a \left(\sin mx \frac{1}{k} e^{-kx} - \frac{m}{k} \left(\frac{1}{k} e^{-kx} \cos mx + \int \frac{m}{k} e^{-kx} \sin mx \right) \right)$

$= a I_{mk} = a \left(\sin mx \frac{1}{k} e^{-kx} - \frac{m}{k^2} e^{-kx} \cos mx - \frac{m^2}{k^2} I_{mk} \right)$

$a I_{mk} = a \left(\sin mx \frac{1}{k} e^{-kx} - \frac{m}{k^2} e^{-kx} \cos mx \right)$

$\frac{1}{1 + \frac{m^2}{k^2}} = \frac{-a \frac{m}{k^2} e^{-kx}}{k^2 + m^2} + c$

$\frac{k^2 + m^2 + am}{k^2 + m^2} = c$

$y = a \frac{(k \sin mx - m \cos mx) + (k^2 + m^2 + am) e^{-kx}}{k^2 + m^2}$

14S

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

15T

No image has yet been uploaded for this question

Solution(s):

2012 Paper I Q15T

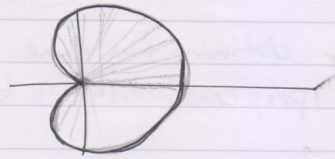
(15)

$$r = 1 + \cos \phi$$

$$x = r \cos \phi$$

$$y = r \sin \phi$$

a)



$$y = r \sin \phi$$

$$y = \sin \phi + \sin \phi \cos \phi$$

$$\frac{dy}{d\phi}$$

$$= \cos \phi + \cos^2 \phi - \sin^2 \phi$$

$$2 \cos^2 \phi + \cos \phi - 1 = 0$$

$$(2 \cos \phi - 1)(\cos \phi + 1) = 0$$

$$\cos \phi = -1$$

$$\cos \phi = \frac{1}{2}$$

minimum

maximum

$$y = (1 + \frac{1}{2}) \cdot \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{4}$$

$$x = (1 + \frac{1}{2}) \cdot \frac{1}{2} = \frac{3}{4}$$

$$\text{maximum } [\frac{3}{4}, \frac{3\sqrt{3}}{4}]$$

b)

$$\int_0^{2\pi} \int_0^{1+\cos \phi} r \, dr \, d\phi$$

$$\int_0^{2\pi} \frac{1}{2} (1 + \cos^2 \phi + 2 \cos \phi) d\phi = \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos^2 \phi + \cos \phi \right) d\phi$$

$$= \pi + \frac{\pi}{2} + 0 = \frac{3\pi}{2}$$

c)

$$(dx)^2 = (dr \cos \phi)^2 + 2 dr \cos \phi \sin \phi d\phi + r^2 \sin^2 \phi d\phi^2$$

$$(dy)^2 = (dr \sin \phi)^2 + 2 dr \sin \phi \cos \phi d\phi + r^2 \cos^2 \phi d\phi^2$$

$$dx^2 + dy^2 = (dr)^2 + r^2 d\phi^2$$

$$\int_0^{2\pi} \sqrt{(dx)^2 + (dy)^2} = \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\phi}\right)^2 + r^2} d\phi$$

$$\sqrt{2} \int_0^{2\pi} \sqrt{1 + \cos \phi} d\phi = \sqrt{2} \int_0^{2\pi} \sqrt{2 \cos^2 \left(\frac{\phi}{2}\right)} d\phi$$

$$= 2 \int_0^{2\pi} \left| \cos \frac{\phi}{2} \right| d\phi = 4 \int_0^{\pi} \cos \frac{\phi}{2} d\phi = 8 \cdot 1 = \underline{\underline{8}}$$

16Y

No image has yet been uploaded for this question

Solution(s):

From user: ar857

① 2012 Paper 1 Q16 4

a) $\frac{\partial f}{\partial x} = \frac{-2}{(1+x^2+y^2)^2} (x, y)$

$$\nabla f(1,0) \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{-2}{(2)^2} \left(\frac{1}{2}, 0 \right) \cdot \left(\frac{4}{5}, \frac{3}{5} \right)$$
$$= \frac{-2}{4} \cdot \frac{4}{5} = -\frac{2}{5}$$

b)

$$\frac{\partial g}{\partial x} = (2x - x(x^2 + y^2)) \cdot e^{-\frac{x^2 + y^2}{2}}$$
$$= x \cdot (2 - x^2 + y^2) e^{-\frac{x^2 + y^2}{2}}$$
$$= 0 \text{ for } x=0 \text{ or } x^2 = 2 + y^2$$
$$\frac{\partial g}{\partial y} = (-2 - x^2 + y^2) y e^{-\frac{x^2 + y^2}{2}}$$
$$= 0 \text{ for } y=0 \text{ or } y^2 = 2 + x^2$$

Steady state points are

$$\begin{aligned} & [0, 0] \\ & [0, \pm\sqrt{2}] \\ & [\pm\sqrt{2}, 0] \end{aligned}$$

c)

$$\frac{\partial^2 g}{\partial x^2} = (2 - x^2 + y^2) e^{-\frac{x^2 + y^2}{2}} - 2x^2 e^{-\frac{x^2 + y^2}{2}} - 2x^2 (2 - x^2 + y^2) e^{-\frac{x^2 + y^2}{2}}$$
$$g_{xx}(0,0) = 2 - 0 - 0 = 2$$
$$g_{yy} = -2$$

since g_{yy} has opposite sign to $g_{xx} \Rightarrow$ saddle point

17R

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

18R

No image has yet been uploaded for this question

Solution(s):

From user: ar857

18) 2012 Paper I Q 18 R

Ben goes $\frac{3}{4}$

Ben stays $\frac{1}{4}$

a) i) $(\frac{3}{4})^{24}$

ii) $(\frac{1}{4})^{24}$

iii) $\frac{3}{4} \cdot 24 = 18$

iv) $\sigma^2 = \frac{3}{4} \cdot \frac{1}{4} \cdot 24 = \frac{9}{2}$
 $\sigma = \frac{3}{\sqrt{2}}$

b) i) $\int_a^b f(x) dx$

ii) $\mu = \langle x \rangle = \int_a^b x f(x) dx$

$\sigma^2 = \int_a^b (x - \mu)^2 f(x) dx = \int_a^b x^2 f(x) dx - (\mu)^2$

From user: ar857

c) $Axe^{-\lambda x}$ for $\lambda > 0$

i) $\int_0^\infty Axe^{-\lambda x} = A \frac{1}{\lambda} \int_0^\infty e^{-\lambda x} = A \frac{1}{\lambda} = 1 \Rightarrow A = \lambda^2$

ii) $\int_0^\infty Ax^2 e^{-\lambda x} = \frac{2}{\lambda} \int_0^\infty Axe^{-\lambda x} = \frac{2}{\lambda} = \mu$

iii) $\sigma^2 = \int_0^\infty Ax^3 e^{-\lambda x} = \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2} = \sigma^2$

iv) $\frac{2}{\lambda} + 2 \frac{\sqrt{2}}{\lambda} = \frac{2+2\sqrt{2}}{\lambda}$

$\phi(X > \mu + 2\sigma) = \int_{\frac{2+2\sqrt{2}}{\lambda}}^\infty \lambda^2 x e^{-\lambda x}$

$= \lambda^2 \left[-\frac{1}{\lambda} x e^{-\lambda x} \right]_{\frac{2+2\sqrt{2}}{\lambda}}^\infty + \frac{\lambda^2}{\lambda} \int_{\frac{2+2\sqrt{2}}{\lambda}}^\infty e^{-\lambda x} dx$

$= \frac{\lambda^2}{\lambda} \frac{2+2\sqrt{2}}{\lambda} e^{-2-2\sqrt{2}} + \frac{\lambda^2}{\lambda^2} e^{-2-2\sqrt{2}}$

$= \left(3 + 2\sqrt{2} \right) e^{-2(1+\sqrt{2})}$

19W*

No image has yet been uploaded for this question
 No solution has yet been submitted for this question.

20Y*

No image has yet been uploaded for this question
 No solution has yet been submitted for this question.