2011 Mathematics (1)

Section A

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Section B

11S

(a) Write down the first four terms of the Taylor expansion of a function $f(x)$ about $x = a$. [4]

(b) Find, by any method, the Taylor expansion about $x = 0$, up to and including the term in x^3, of the following functions:

(i)

$$\frac{1}{(x^2+9)^{1/2}},$$

[6]

(ii)

$$\ln[(2+x)^3],$$

[4]

(iii)

$$e^{\sin x}.$$

[6]

12X

The Dean of Porterhouse (the oldest and most famous of the colleges of the University of Carrbridge) leads a service in Chapel every Sunday. The length s of each service, in minutes, is exponentially distributed with a mean of $\bar{s}$ minutes.

(a) Write down the probability density function for s.

Unfortunately, some of the Dean's services are being interrupted as a result of an electrical fault in the chapel organ. This fault causes one of the organ's pipes to spontaneously emit a loud sound t minutes after the beginning of the service. It is found that t is exponentially distributed with mean $\bar{t}$ minutes and is independent of s.

(b) Draw a pair of axes at right-angles to each other labelling one s and one t. Indicate on this diagram the region of the (s, t)-plane in which the service is **not** interrupted by the organ.

(c) The probability of being in some region of this plane is the double integral of the product of the density functions for s and t integrated over the region. Explain in words why this is so.

(d) Calculate the *probability* (as a function of $\bar{s}$ and $\bar{t}$) that the service is **not** interrupted by the organ.

Define the random variable r to be equal to "-1" if the organ does not interrupt the service, and equal to "the number of minutes of the service which are remaining, at the moment the organ makes a noise" if the organ interrupts the service.

(e) On the same diagram as before, indicate the region of the (s, t)-plane in which $r > r_0$, where r_0 is a positive constant.

(f) Calculate the *probability* (as a function of r_0, $\bar{s}$ and $\bar{t}$) that r is greater than r_0 minutes, again assuming that r_0 is a positive constant.

(g) Consider the answer for $P(r > r_0)$ that you have found in (f), and comment on whether it seems sensible in each of the following limits:

(i) $r_0 \to \infty$, (ii) $r_0 \to 0$ in the case $\bar{s} = \bar{t}$.

Suppose that it is 6:54pm, that the service was interrupted by the organ at 6:22pm, and that the Dean is *still* talking. The Fellows are getting hungry, and are wondering how many *more* minutes, m, they are going to have to stay sitting in Chapel until the service ends.

(h) State (or calculate) the expected value of m.

13Y

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14R

This question involves solving the differential equation

$$\sqrt{3}\frac{dy}{dx} + y = 4\sin x$$

by Fourier methods.

(a) Write down a Fourier series expansion of an arbitrary periodic function which has period 2π. [3]

(b) Suppose that $y(x)$ has such an expansion. Substitute the Fourier series expansion into the differential equation in order to obtain a constraint on its coefficients. [2]

(c) Why may we equate the coefficients of $\sin(mx)$ in this constraint (for each integer m)? You may also equate the coefficients of $\cos(mx)$. [2]

(d) By equating coefficients as described in (c), find all of the coefficients of the Fourier series expansion of $y(x)$. [5]

(e) Thus, write down the explicit form of the periodic solution $y(x)$ in only one term. [1]

(f) Sketch $y(x)$ for $0 \leqslant x \leqslant 2\pi$, clearly displaying maxima and minima. [3]

(g) Use Parseval's theorem to evaluate $\int_0^{2\pi} \{y(x)\}^2 dx$. [2]

(h) Check your answer to (g) by performing the integral explicitly. [2]

15S

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16T

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17Z

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18Z

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19R*

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20X*

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