2011 Mathematics (2)

Section A

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Section B

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19X*

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20R*

Consider the functions $L_n(x)$ defined by

$$L_n(x) = \frac{e^x}{n!}\,\frac{d^n}{dx^n}\,\left(x^n e^{-x}\right),$$

for non-negative integers n.

(a) Show that L_n is a polynomial of degree n and find the coefficients of x^{10} and x^9 of $L_{10}(x)$ in terms of factorials. [6]

(b) Let $v = x^n\,e^{-x}$. Verify that

$$x\,\frac{dv}{dx} = (n-x)v,$$

and differentiate this identity $n+1$ times to show that L_n satisfies the differential equation

$$x\,\frac{d^2 L_n}{dx^2} + (1-x)\,\frac{dL_n}{dx} + nL_n = 0.$$

[8]

(c) The previous equation can be rewritten as

$$e^x\,\frac{d}{dx}\left(xe^{-x}\,\frac{dL_n}{dx}\right) + nL_n = 0.$$

Hence, for any non-negative integers n, m,

$$\begin{aligned} L_m\,e^x\,\frac{d}{dx}\left(xe^{-x}\,\frac{dL_n}{dx}\right) + nL_mL_n = 0, \\ L_n\,e^x\,\frac{d}{dx}\left(xe^{-x}\,\frac{dL_m}{dx}\right) + mL_mL_n = 0. \end{aligned}$$

Use these results to derive the orthogonality relation

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = 0$$

for $m \neq n$. [6]