2011 Mathematics (2)

Section A

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Section B

11R

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Solution(s):

From user: ar857

(11) 2011 Paper 2 Q11

a) i) $\oint_{T_1} (3v + w) \cdot dx = \oint_{T_1} (3v)\cdot dx + \oint_{T_1} w \cdot dx$

A sensible split

$\oint_{T_1} (3v)\cdot dx = 0$ because v is a conservative field since it is a gradient of ϕ

$$\oint_{T_1} (3v+w)\cdot dx = \int_0^{2\pi} (0,0,\cos\phi)\cdot\left(-\sin\phi, \frac{d}{d\phi}\left(e^{-\cos^2\phi}\right), \cos\phi\right) d\phi$$

$$= \int_0^{2\pi} \cos^2\phi \, d\phi = \frac{1}{2}\cdot 2\pi = \pi$$

ii) $\int_{T_2} (\nabla \wedge v)\cdot dx = 0$ since curl of conservative field $= 0$

iii) $\int_{T_2} \nabla\cdot\left[\nabla\wedge\left(|v|^2 w\right)\right] ds = 0$

$\nabla\cdot(\nabla\wedge f(\mathbf{x})) = 0$

because $\nabla\cdot(\nabla\wedge f(x,y,z)) = \frac{\partial}{\partial x}(g_1(y,z)) + \frac{\partial}{\partial y}(g_2(x,z)) + \frac{\partial}{\partial z}(g_3(x,y)) = 0$

b) i) $\int_S (\nabla\cdot v)\, d\mathbf{S} = \int_S (\nabla\cdot v)\,\mathbf{n}\, dS$

$$\nabla\cdot v = \nabla\cdot(\nabla\phi) = \nabla\cdot\left(\nabla\left(e^{x^2+y^2+z^2}\right)\right) = \nabla\cdot\left(2e^{x^2+y^2+z^2}(x,y,z)\right)$$

$$= 6e^{x^2+y^2+z^2} + 4e^{x^2+y^2+z^2}(x^2+y^2+z^2)$$

$dS = a^2 \sin\sigma \, d\sigma \, d\phi \qquad a = 1$

$x = \cos\phi\sin\sigma \quad y = \sin\sigma\sin\phi \quad z = \cos\sigma$

$n = (\cos\phi\sin\sigma, \sin\phi\sin\sigma, \cos\sigma)$

S is closed, the surface with outward normal; v is conservative so expect $\int_S = -\int_S$?

$$\int_S (\nabla\cdot v)\, d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi/2} 6e^1 + 4e^1(1)\,(\cos\phi\sin\sigma^2, \sin\phi\sin\sigma^2, \cos\sigma\sin\sigma)\, 1\, d\sigma\, d\phi$$

$$= \int_0^{2\pi}\int_0^{\pi/2} 10e\,(\cos\phi\sin\sigma^2, \sin\phi\sin\sigma^2, \cos\sigma\sin\sigma)\, d\sigma\, d\phi$$

$$= 2\pi\int_0^{\pi/2} 10e\left(0, 0, \tfrac{1}{2}\sin 2\sigma\right) d\sigma = 10\pi e\cdot\tfrac{1}{2}(-\cos\pi + \cos 0)\,k = 10\pi e k = (0,0,10\pi e)$$

sloppy

ii) $\int_S (\nabla\cdot v)\, dS =$

$$\int_{\pi/2}^{\pi} 10e\pi(0,0,\sin 2\sigma) = 10e\pi\cdot(0-1)k = (0,0,-10e\pi)$$

12S

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Solution(s):

From user: ar857

(12) 2011 Paper 2 Q12

a) $x = r\cos\phi \sin\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\theta$

b) $$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$= r^2 \cdot \left[(\cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi)^2 + (\cos\phi\sin\theta\, d\phi + \cos\theta\sin\phi\, d\theta)^2 + (0 - \sin\theta\, d\theta)^2 \right]$$

$$= r^2 \cdot \left(\cos^2\theta\cos^2\phi (d\theta)^2 - \cancel{2\cos\theta\cos\phi\sin\theta\sin\phi\, d\phi\, d\theta} + \sin^2\theta\sin^2\phi (d\phi)^2 + \cos^2\phi (\sin\theta)^2 (d\phi)^2 + (\cos\theta)^2(\sin\phi)^2(d\theta)^2 + \cancel{2\cos\theta\cos\theta\sin\phi\sin\theta\, d\theta\, d\phi} + \sin^2\theta (d\theta)^2 \right)$$

$$= r^2 \cdot \left(\cos^2\theta\, d\theta^2 + \sin^2\theta\, d\theta^2 + \sin^2\theta (d\phi)^2 \right)$$

$$= r^2 \cdot \left((d\theta)^2 + \sin^2\theta (d\phi)^2 \right)$$

$$\delta s^2 = r^2 (\delta\theta^2 + \sin^2\theta\, \delta\phi^2)$$ for sufficiently small $\delta\phi$ and $\delta\theta$

c) plane: $Ax + By + Cz = 0$

sphere $x^2 + y^2 + z^2 = R^2 \iff r^2 = R^2$

plane $\underbrace{AR}_{l}\cos\phi\sin\theta + \underbrace{BR}_{n}\sin\phi\sin\theta + \underbrace{CR}_{m}\cos\theta = 0$

d)

A: $l\cos 0 \sin\frac{\pi}{2} + n\sin 0 \sin\frac{\pi}{2} + n\cos\frac{\pi}{2} = 0$

$l = 0$

B: $m\sin\phi_0\sin\theta_0 + n\cos\theta_0 = 0$

$$-n = \frac{m\sin\phi_0\sin\theta_0}{\cos\theta_0}$$

$$-\frac{n}{m} = \tan\theta_0\sin\phi_0$$

equivalently from original equation $-\frac{n}{m} = \tan\theta\sin\phi$

$$-\frac{n}{m} = \tan\theta_0\sin\phi_0 = \tan\theta\sin\phi$$

13Z

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Solution(s):

From user: ar857

(13) $\frac{1}{\cosh x} < \frac{1}{\sinh x}$ 2011 Paper 2 Q13

a) $\frac{2}{e^x+e^{-x}} < \frac{2}{e^x-e^{-x}} \quad |:2$

$\frac{e^x}{e^{2x}+1} < \frac{e^x}{e^{2x}-1} \quad |\cdot(e^{2x}-1)(e^{2x}+1)\frac{1}{e^x}$ since $x>0$ $e^x>1$ and $e^{2x}>1$ and $e^{2x}-1>0$

$e^{2x}-1 < e^{2x}+1$

$-1<1$

therefore $\frac{1}{\cosh x} < \frac{1}{\sinh x}$ for $x>0$

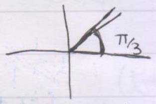
$\frac{1}{\sinh x} < \frac{\cosh}{\sinh x}$ $\quad \frac{2e^x}{e^{2x}-1} < \frac{e^x\cdot(e^x+e^{-x})}{e^{2x}-1}$ → true for $x>0$

$1 < \frac{e^x+e^{-x}}{2}$

$1 < \cosh x$ true for all x except $x=0$

therefor $\frac{1}{\cosh x} < \frac{1}{\sinh x} < \coth x$ for $x>0$

b) $\nabla f(x,y) = \nabla(e^{xy}) = (ye^{xy}, xe^{xy})$



$\nabla f(-2,0) = (0,-2)$

$\nabla f(-2,0)\cdot\hat{u} = (0,-2)\cdot(\frac{1}{2},\frac{\sqrt{3}}{2}) = -\sqrt{3}$ $\quad (\frac{1}{2},\frac{\sqrt{3}}{2})$

c) $f(x,y)=e^{xy}$

$\frac{\partial f}{\partial y} = xe^{xy} = 0$ for $x=0$

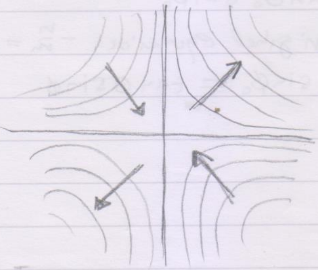
$\frac{\partial f}{\partial x} = ye^{xy} = 0$ for $y=0$ $\quad$ St. point at $(0,0)$

$\frac{\partial^2 f}{\partial x^2} = y^2e^{xy}$ $\quad (\frac{\partial f}{\partial y})^2 = x^2e^{xy}$ $\quad \frac{\partial^2 f}{\partial y\partial x} = yxe^{xy}$

~~tan 60~~

tan undefined at $\frac{\pi}{2} = 90^\circ$

saddle at $(0,0)$



14Y

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15T

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Solution(s):

From user: ar857

15 2011 Paper 2 Q15

a) $\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} = 0$ $\quad \lambda^2 + \gamma\lambda = 0$

$\lambda \cdot (\lambda + \gamma) = 0 \quad \lambda_1 = 0 \quad \lambda_2 = -\gamma$

$x = C_1 + C_2 e^{-\gamma t}$

$\frac{d^2z}{dt^2} + \gamma \frac{dz}{dt} + g = 0 \qquad \frac{d^2z}{dt^2} + \gamma\left(\frac{dz}{dt} + \frac{g}{\gamma}\right) = 0$

$z = C_3 + C_4 e^{-\gamma t} - \frac{g}{\gamma} t$

$V_{0x} = -\gamma C_2$

$C_2 = -\frac{V_{0x}}{\gamma}$

$C_1 = -C_2 = \frac{V_{0x}}{\gamma}$

$V_{0z} = -\gamma C_4 - \frac{g}{\gamma}$

$C_4 = \frac{V_{0z} + \frac{g}{\gamma}}{-\gamma}$

$C_3 = \frac{V_{0z} + \frac{g}{\gamma}}{\gamma}$

$x(t) = \frac{V_{0x}}{\gamma}\left(1 - e^{-\gamma t}\right)$

$z(t) = \frac{V_{0z}}{\gamma}\left(1 - e^{-\gamma t}\right) + \frac{g}{\gamma^2}\left(1 - e^{-\gamma t}\right) - \frac{g}{\gamma} t$

b)

$z = z(x)$

$z = \frac{V_{0z}}{\gamma}\left(\frac{x\gamma}{V_{0x}}\right) + \frac{g}{\gamma^2}\left(\frac{x\gamma}{V_{0x}}\right) - \frac{g}{\gamma} t$

$= \frac{x}{V_{0x}} \cdot \left(V_{0z} + \frac{g}{\gamma}\right) - \frac{g}{\gamma} t$

$t = \left(\frac{-x + \frac{V_{0x}}{\gamma}}{\frac{V_{0x}}{\gamma}}\right) \cdot \frac{-1}{\gamma}$

16T

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Solution(s):

From user: ar857

2011 Paper 2 Q16

16 a) $\int_{-1/2}^{1/2} \sqrt{\frac{9 - 4x^2}{36(1-x^2)}}\, dx \qquad x = \sin\sigma \quad dx = \cos\sigma \quad -\frac{1}{2} = \sin\sigma$

$= \int_{-\pi/6}^{\pi/6} \sqrt{\frac{9 - 4\sin\sigma^2}{36\cos\sigma^2}} \cos\sigma\, d\sigma = \int_{-\pi/6}^{\pi/6} \sqrt{\frac{9 - 4\sin\sigma^2}{9 \cdot 4}}\, d\sigma$

$= 2 \cdot \int_0^{\pi/6} \frac{1}{2} \sqrt{1 - \frac{4}{9}\sin\sigma^2}\, d\sigma = \int_0^{\pi/6} \sqrt{1 - \left(\frac{2}{3}\right)^2 \sin\sigma^2}\, d\sigma = E\left(\frac{2}{3}, \frac{\pi}{6}\right)$

From user: ar857

b) $L=\int\sqrt{dx^2+dy^2}$ $\quad x = a\sin\sigma$

$dx = a\cos\sigma\, d\sigma$

$dy = -b\sin\sigma\, d\sigma$

$L=\int_0^{2\pi}\sqrt{a^2\cos\sigma^2+b^2\sin\sigma^2}\,d\sigma$

$e^2 = \frac{a^2-b^2}{a^2} = 1-\frac{b^2}{a^2}$

$L=\int_0^{2\pi}\sqrt{a^2 - a^2\sin\sigma^2+b^2\sin\sigma^2}\,d\sigma$

$=\int_0^{2\pi} a\sqrt{1-\sin\sigma^2+\frac{b^2}{a^2}\sin\sigma^2}\,d\sigma = a\int_0^{2\pi}\sqrt{1-e^2\sin\sigma^2}\,d\sigma$

$=4a\int_0^{\pi/2}\sqrt{1-e^2\sin\sigma^2}\,d\sigma = 4aE(e,\pi/2)$

c) $L = a\int_0^{2\pi} d\sigma\,\left[1-\frac12 e^2\sin\sigma^2\right]$

$= a\int_0^{2\pi} d\sigma\,\left[1-\frac12 e^2\left(\frac{1-\cos 2\sigma}{2}\right)\right] =$

$= a\int_0^{2\pi} d\sigma\,\left[1-\frac14 e^2+\frac14 e^2\cos 2\sigma\right]$

$= a(2\pi)\cdot\left(1-\frac14 e^2+0\right)$

$= 2\pi a\cdot\left(1-\frac14+\frac{b^2}{4a^2}\right) = 2\pi a\left(\frac34+\frac{b^2}{4a^2}\right) = \frac{\pi}{2a}\left(3a^2+b^2\right)$

17X

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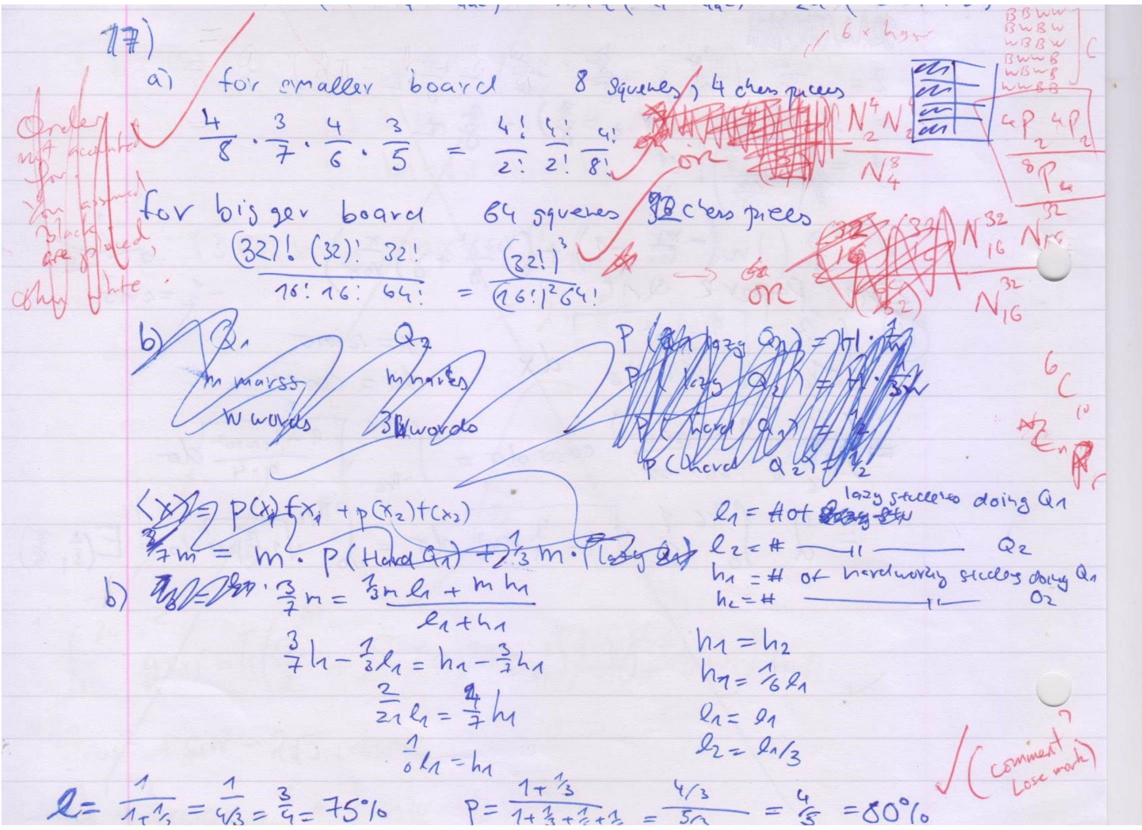
Solution(s):

From user: ar857



17) a) for smaller board 8 squares, 4 chess pieces

$$\frac{4}{8}\cdot\frac{3}{7}\cdot\frac{4}{6}\cdot\frac{3}{5} = \frac{4!}{2!}\frac{4!}{2!}\frac{4!}{8!}$$

for bigger board 64 squares, 32 chess pieces

$$\frac{(32)!\,(32)!\,32!}{16!\,16!\,64!} = \frac{(32!)^3}{(16!)^2 64!}$$

Order not accounted for / You assumed black are placed after white

b) Q_1 m marks, W words; Q_2 m marks, 3W words

$\langle x\rangle = p(x_1)f(x_1) + p(x_2)f(x_2)$

$\frac{3}{7}m = m\cdot P(\text{Hard }Q_1) + \frac{1}{3}m\cdot P(\text{lazy }Q_1)$

b) $\frac{3}{7}m = \frac{\frac{1}{3}m\,l_1 + m\,h_1}{l_1+h_1}$

$\frac{3}{7}h - \frac{1}{3}l_1 = h_1 - \frac{3}{7}h_1$

$\frac{2}{21}l_1 = \frac{4}{7}h_1$

$\frac{1}{6}l_1 = h_1$

l_1 = # of lazy students doing Q_1

l_2 = # — " — Q_2

h_1 = # of hardworking students doing Q_1

h_2 = # — " — Q_2

$h_1 = h_2$

$h_1 = \frac{1}{6}l_1$

$l_1 = l_1$

$l_2 = l_1/3$

$l = \frac{1}{1+\frac{1}{3}} = \frac{1}{4/3} = \frac{3}{4} = 75\%$ $\quad P = \frac{1+\frac{1}{3}}{1+\frac{1}{3}+\frac{1}{6}+\frac{1}{6}} = \frac{4/3}{5/3} = \frac{4}{5} = 80\%$

(comment lose marks)

18Y

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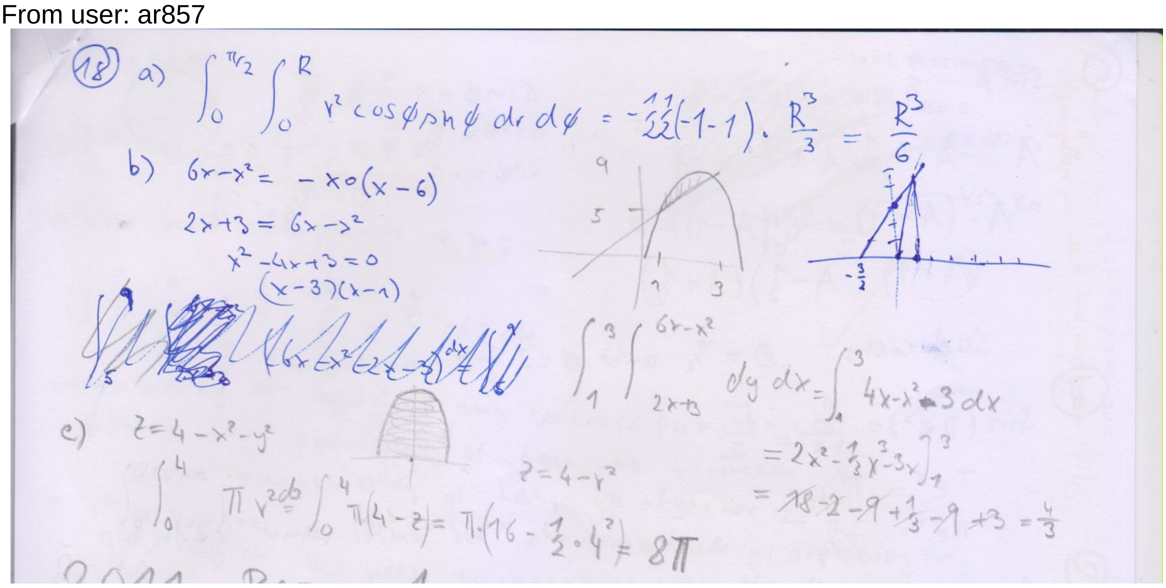
Solution(s):

From user: ar857



18) a) $\int_0^{\pi/2}\int_0^R r^2\cos\phi\sin\phi\,dr\,d\phi = -\frac{1}{2}\frac{1}{2}(-1-1)\cdot\frac{R^3}{3} = \frac{R^3}{6}$

b) $6x-x^2 = -x(x-6)$

$2x+3 = 6x-x^2$

$x^2-4x+3=0$

$(x-3)(x-1)$

$\int_1^3\int_{2x+3}^{6x-x^2} dy\,dx = \int_1^3 4x-x^2-3\,dx$

$= 2x^2-\frac{1}{3}x^3-3x\Big]_1^3$

$= 18-2-9+\frac{1}{3}-9+3 = \frac{4}{3}$

c) $z = 4-x^2-y^2$

$\int_0^4 \pi r^2 dz = \int_0^4 \pi(4-z) = \pi\cdot(16-\frac{1}{2}\cdot 4^2) = 8\pi$ $\quad z = 4-r^2$

19X*

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20R*

Consider the functions $L_n(x)$ defined by

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right),$$

for non-negative integers n.

(a) Show that L_n is a polynomial of degree n and find the coefficients of x^{10} and x^9 of $L_{10}(x)$ in terms of factorials. [6]

(b) Let $v = x^n e^{-x}$. Verify that

$$x\frac{dv}{dx} = (n-x)v,$$

and differentiate this identity $n+1$ times to show that L_n satisfies the differential equation

$$x\frac{d^2 L_n}{dx^2} + (1-x)\frac{dL_n}{dx} + nL_n = 0.$$

[8]

(c) The previous equation can be rewritten as

$$e^x\frac{d}{dx}\left(xe^{-x}\frac{dL_n}{dx}\right) + nL_n = 0.$$

Hence, for any non-negative integers n, m,

$$L_m e^x\frac{d}{dx}\left(xe^{-x}\frac{dL_n}{dx}\right) + nL_mL_n = 0,$$
$$L_n e^x\frac{d}{dx}\left(xe^{-x}\frac{dL_m}{dx}\right) + mL_mL_n = 0.$$

Use these results to derive the orthogonality relation

$$\int_0^\infty e^{-x}L_m(x)L_n(x)dx = 0$$

for $m \neq n$. [6]

Solution(s):

From user: cgl20

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

(a) By Leibniz

$$L_n(x) = \frac{e^x}{n!}\sum_{a=0}^{n} \frac{d^{(n-a)}}{dx^{(n-a)}}\left(x^n\right)\frac{d^a}{dx^a}\left(e^{-x}\right)\binom{n}{a} = \frac{\cancel{e^x}}{\cancel{n!}}\sum_{a=0}^{n}\frac{\cancel{n!}}{a!}x^a\,\cancel{e^{-x}}(-1)^a\binom{n}{a}$$

$$= \sum_{a=0}^{n}(-x)^a\frac{n!}{a!\,a!\,(n-a)!}$$ which is a polynomial in x of degree n.

$$\therefore\quad L_{10}(x) = (-x)^{10}\frac{10!}{10!\,10!\,0!} + (-x)^9\frac{10!}{9!\,9!\,1!} + \cdots = \frac{1}{10!}x^{10} - \frac{10!}{(9!)^2}x^9 + \cdots$$

(b) $v = x^n e^{-x}$. $\therefore\ x\frac{dv}{dx} = x\left(nx^{n-1}e^{-x} - x^n e^{-x}\right) = (n-x)x^n e^{-x} = (n-x)v.$ QED

We see that $L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(v)$. $x\frac{dv}{dx} = (n-x)v \Rightarrow \frac{d^{n+1}}{dx^{n+1}}\left(x\frac{dv}{dx}\right) = \frac{d^{n+1}}{dx^{n+1}}(n-x)v$

$$\Rightarrow \left(\frac{d^{n+1}}{dx^{n+1}}\left(\frac{dv}{dx}\right)\right)x + \binom{n+1}{1}\frac{d^n}{dx^n}\left(\frac{dv}{dx}\right)\cdot 1 = \left(\frac{d^{n+1}}{dx^{n+1}}v\right)(n-x) + \binom{n+1}{1}\frac{d^n v}{dx^n}\cdot(-1)$$

$$\Rightarrow \frac{(n+2)!}{e^x}L_{n+2}\,x + (n+1)\frac{(n+1)!}{e^x}L_{n+1} = (n-x)\frac{(n+1)!}{e^x}L_{n+1} - (n+1)\frac{n!}{e^x}L_n$$

$$\Rightarrow (n+2)xL_{n+2} + (1+x)L_{n+1} + L_n = 0. \qquad (\ast)$$

However,

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(v) \Rightarrow \frac{dL_n}{dx} = \frac{e^x}{n!}\frac{d^{n+1}}{dx^{n+1}}v + \frac{e^x}{n!}\frac{d^n}{dx^n}v = (n+1)L_{n+1} + L_n$$

$$\& \quad \frac{d^2L_n}{dx^2} = (n+1)\frac{dL_{n+1}}{dx} + \frac{dL_n}{dx} = (n+1)\left((n+2)L_{n+2} + L_{n+1}\right) + (n+1)L_{n+1} + L_n = (n+1)(n+2)L_{n+2} + 2(n+1)L_{n+1} + L_n \quad (\ast\ast)$$

$$\therefore\ (1-x)\frac{dL_n}{dx} + nL_n = (1-x)\left((n+1)L_{n+1} + L_n\right) + nL_n = (1-x)(n+1)L_{n+1} + (n+1-x)L_n$$

(by $\ast$)

$$= (1-x)(n+1)L_{n+1} + (n+1)\left[-(n+2)xL_{n+2} - (1+x)L_{n+1}\right] - xL_n$$

$$= -x(n+2)(n+1)L_{n+2} - 2(n+1)xL_{n+1} - xL_n$$

$$= -x\frac{d^2L_n}{dx^2} \qquad (\text{by } (\ast\ast))$$

$$\therefore\ x\frac{d^2L_n}{dx^2} + (1-x)\frac{dL_n}{dx} + nL_n = 0.$$ QED

(c) Define $I_{mn} = \int_0^\infty e^{-x}L_m(x)L_n(x)dx$.

Clearly $I_{mn} = I_{nm}$. We are asked to show that $I_{mn} = 0$ when $m \neq n$.
Since I_{mn} is symmetric $m \leftrightarrow n$ and we are only interested in $m \neq n$ we may assume, without loss of generality, that $n \neq 0$. Assuming, therefore, that $m \neq n$ & $n \neq 0$ we may say:

$$L_m e^x \frac{d}{dx}\left(xe^{-x}\frac{dL_n}{dx}\right) + nL_mL_n = 0, \quad (A)$$
$$L_n e^x \frac{d}{dx}\left(xe^{-x}\frac{dL_m}{dx}\right) + mL_mL_n = 0. \quad (B)$$

$$I_{mn} = -\frac{1}{n}\int_0^\infty L_m\frac{d}{dx}\left(xe^{-x}\frac{dL_n}{dx}\right)dx \qquad (\text{by } (A))$$

$$= -\frac{1}{n}\left\{\left[L_m x\cancel{e^{-x}\frac{dL_n}{dx}}\right]_0^\infty - \int_0^\infty \frac{dL_m}{dx}\,x\,e^{-x}\frac{dL_n}{dx}\,dx\right\} \qquad (\text{by parts})$$

$$= \frac{1}{n}\int_0^\infty \frac{dL_n}{dx}\,xe^{-x}\frac{dL_m}{dx}\,dx$$

$$= \frac{1}{n}\left\{\left[L_n x\cancel{e^{-x}\frac{dL_m}{dx}}\right]_0^\infty - \int_0^\infty L_n\frac{d}{dx}\left(xe^{-x}\frac{dL_m}{dx}\right)dx\right\} \qquad (\text{by parts})$$

$$= -\frac{1}{n}\int_0^\infty -me^{-x}L_mL_n\,dx \qquad (\text{by } (B))$$

$$= \frac{m}{n}I_{mn} \Rightarrow I_{mn}\left(1 - \frac{m}{n}\right) = 0 \Rightarrow I_{mn} = 0 \text{ since } m \neq n.$$