

2008 Mathematics (1)

This pdf was generated from questions and answers contributed by members of the public to Christopher Lester's tripos/example-sheet solution exchange site <http://cgl20.user.srcf.net/>. Nothing (other than raven authentication) prevents rubbish being uploaded, so this pdf comes with no warranty as to the correctness of the questions or answers contained. Visit the site, vote, and/or supply your own content if you don't like what you see here.

This pdf had url <http://cgl20.user.srcf.net/camcourse/paperpdf/20?withSolutions=1>.

This pdf was creted on Tue, 23 Apr 2024 07:32:26 +0000.

Section A

1

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

2

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

3

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

4

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

5

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

6

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

7

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

8

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

9

No image has yet been uploaded for this question
No soution has yet been submitted for this question.

10

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

Section B

11Y

No image has yet been uploaded for this question

Solution(s):

From user: cgl20

$$K \underline{r} = \underline{r} - (\underline{n} \cdot \underline{r}) \underline{n}, \quad |\underline{n}| = 1.$$

$$\begin{aligned} K^2 \underline{r} &= (\underline{r} - (\underline{n} \cdot \underline{r}) \underline{n}) - (\underline{n} \cdot (\underline{r} - (\underline{n} \cdot \underline{r}) \underline{n})) \underline{n} \\ &= \underline{r} - (\underline{n} \cdot \underline{r}) \underline{n} - (\underline{n} \cdot \underline{r}) \underline{n} + (\underline{n} \cdot \underline{n}) (\underline{n} \cdot \underline{r}) \underline{n} \\ &= \underline{r} - 2(\underline{n} \cdot \underline{r}) \underline{n} + (\underline{n} \cdot \underline{n}) (\underline{n} \cdot \underline{r}) \underline{n} \\ &= \underline{r} \quad \underline{\text{QED.}} \end{aligned}$$

Suppose $\underline{a} \parallel \underline{n}$, ie $\underline{a} = \lambda \underline{n}$.

$$\begin{aligned} \text{Then } K \underline{a} &= \lambda \underline{n} - (\underline{n} \cdot \lambda \underline{n}) \underline{n} \\ &= \lambda \underline{n} - \lambda \underline{n} \\ &= \underline{0} \quad \underline{\text{QED}} \end{aligned}$$

Suppose $\underline{b} \cdot \underline{n} = 0$.

$$\text{Then } K \underline{b} = \underline{b} - (\underline{n} \cdot \underline{b}) \underline{n} = \underline{b} \quad \underline{\text{QED.}}$$

Any vector \underline{r} can be decomposed into parts parallel & perp to \underline{n} :

$$\underline{r} = \underline{r}_{\parallel} + \underline{r}_{\perp} \quad \text{where } \underline{r}_{\parallel} = \lambda \underline{n}, \quad \underline{r}_{\perp} \cdot \underline{n} = 0.$$

$$\begin{aligned} \therefore K \underline{r} &= K \underline{r}_{\parallel} + K \underline{r}_{\perp} \quad (\text{by linearity of } K) \\ &= \underline{0} + \underline{r}_{\perp} \quad (\text{by last two results}) \\ &= \underline{r}_{\perp} \quad \underline{\text{QED.}} \end{aligned}$$

Now let $\underline{n} = (1, 1, 1)/\sqrt{3}$

$$\underline{e}_1, \underline{e}_2 \text{ satisfy } \underline{e}_1 \cdot \underline{n} = \underline{e}_2 \cdot \underline{n} = 0.$$

$$\underline{e}_1 \propto (1, 1, ?) \quad (\text{since equally inclined to } x \text{ \& } y \text{ axes})$$

$$\therefore \underline{e}_1 \cdot \underline{n} = 0 \Rightarrow \underline{e}_1 = (1, 1, -2)/\sqrt{6}.$$

$$\therefore \underline{e}_2 = (\text{by inspection}) = (1, -1, 0)/\sqrt{2}.$$

$$\text{Suppose: } \underline{r} = \lambda \underline{n} + \mu_1 \underline{e}_1 + \mu_2 \underline{e}_2. \quad \underline{r} = (x, y, z)$$

Since $\underline{n}, \underline{e}_1, \underline{e}_2$ orthogonal, can find λ, μ_1, μ_2 by dotting

$$\Rightarrow \mu_1 = \underline{r} \cdot \underline{e}_1 = (x+y-2z)/\sqrt{6}. \quad (\text{ignoring centre translation})$$

$$\mu_2 = \underline{r} \cdot \underline{e}_2 = (x-y)/\sqrt{2}.$$

Circles in plane $z=0$ could be characterised by being points for which $x^2+y^2 = r^2$ (fixed) and $z=0$.

$$\text{For these } \left\{ \begin{array}{l} \mu_1 = (x+y)/\sqrt{6} \\ \mu_2 = (x-y)/\sqrt{2} \end{array} \right\} \text{ from above.}$$

Evidently, major & minor axes are @ 45° to \underline{e}_1 & \underline{e}_2 , and the ratio of their lengths is $\sqrt{6}/\sqrt{2} = \sqrt{3}$.

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

13Z

No image has yet been uploaded for this question

Solution(s):

From user: ar857

2008 I

13 2

$$\begin{aligned} a) f(x) &= \ln(1+x) & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x} & f'(0) &= 1 \\ f''(x) &= -\frac{1}{(1+x)^2} & f''(0) &= -1 \\ f'''(x) &= \frac{2}{(1+x)^3} & f'''(0) &= 2 \\ f^{(4)}(x) &= -\frac{6}{(1+x)^4} & f^{(4)}(0) &= -6 \\ f^{(5)}(x) &= \frac{24}{(1+x)^5} & f^{(5)}(0) &= 24 \end{aligned}$$

$$\begin{aligned} \ln(1+x) &= 0 + x + \frac{x^2}{2} \cdot 2 + \frac{x^3}{3!} \cdot (-1) + \frac{x^4}{4!} \cdot 24 + \dots \\ \ln(1-x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots + (-1)^{n-1} \frac{x^n}{n} + \dots \\ \ln(1-x) &= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots \\ \ln(1-x) &= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots + \frac{x^n}{n} + \dots\right) \end{aligned}$$

$$\begin{aligned} \ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) = x + x - \frac{x^2}{2} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^4}{4} + \dots \\ &= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots + \frac{x^n}{n} + \dots\right) \end{aligned}$$

$$\begin{aligned} b) (1-e^x) \cdot (1+\frac{x}{3})^{-3} + \ln(1-x) \\ (1-e^x) &= 1 - \left(1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots\right) = -x - \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{24} - \dots \\ (1+\frac{x}{3})^{-3} &= 1 - x + \frac{12}{2} \cdot \frac{x^2}{9} - \frac{60}{6} \cdot \frac{x^3}{27} + \frac{360}{24} \cdot \frac{x^4}{81} - \dots \\ \ln(1-x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

$$\begin{aligned} (1-e^x) \cdot (1+\frac{x}{3})^{-3} + \ln(1-x) &= \left(-x + x^2 - \frac{x^2}{2} + \frac{2}{3}x^3 + \frac{x^3}{2} - \frac{x^3}{6} - \frac{x^4}{24} + \frac{x^4}{6} - \frac{x^4}{3} + \frac{10}{24}x^4\right) \\ &+ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \end{aligned}$$

$$\begin{aligned} &= 0 + 0 + 0 + x^4 \cdot \left(\frac{10}{24} - \frac{1}{3} + \frac{1}{6} - \frac{1}{24} - \frac{1}{4}\right) = x^4 \cdot \left(\frac{1}{24} - \frac{1}{8}\right) \\ &= \frac{x^4}{24} - \frac{x^4}{8} = \frac{x^4}{24} - \frac{3x^4}{24} = -\frac{2x^4}{24} = -\frac{x^4}{12} \end{aligned}$$

$$\begin{aligned} c) (1+ax)(1+bx)^{-1} \cdot \ln(1+x) \\ = (1+ax) \cdot (1-bx+b^2x^2-b^3x^3) \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) \\ = (1-bx+b^2x^2-b^3x^3+a-abx^2+ab^2x^3-ab^3x^4) \cdot \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) \end{aligned}$$

$$0 = x^2 \cdot (-\frac{1}{2} - b + a) \Rightarrow a = b + \frac{1}{2}$$

$$0 = x^3 \cdot \left(\frac{1}{3} + \frac{b}{2} + b^2 - \frac{a}{2} - ab\right) = \frac{1}{3} + \frac{b}{2} + b^2 - \frac{b}{2} - \frac{1}{4} - b^2 - \frac{b}{2} = \frac{1}{12} - \frac{b}{2} = 0$$

$$\Rightarrow b = \frac{1}{6} \quad a = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}$$

$$\begin{aligned} x^4 \cdot \left(-\frac{1}{4} - \frac{b}{3} - \frac{b^2}{2} - b^3 + \frac{a}{3} + \frac{ab}{2} + ab^2\right) &= x^4 \cdot \left(-\frac{1}{4} - \frac{1}{18} - \frac{1}{72} - \frac{1}{216} + \frac{2}{9} - \frac{1}{18} + \frac{1}{54}\right) \\ &= x^4 \cdot \left(-\frac{9}{36} - \frac{2}{36} - \frac{0.5}{36} - \frac{1}{36} + \frac{8}{36} + \frac{2}{36} + \frac{1}{36}\right) = \frac{-1}{36} \end{aligned}$$

14R

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

15X

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

16Z

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

17T

No image has yet been uploaded for this question

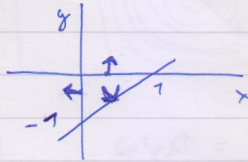
Solution(s):

From user: ar857

2008 I 7 T

a) i) $f(x, y) = xy(1-x+y)$

$f=0$ at $x=0$ at $y=0$ and $y=x-1$



ii) $\frac{\partial f}{\partial x} = y(1-x+y) - xy$

$\frac{\partial f}{\partial y} = x(1-x+y) + xy$

at $(\frac{1}{2}, 0) = (0, \frac{1}{4})$

$(\frac{1}{2}, -\frac{1}{2}) = (\frac{1}{4}, -\frac{1}{4})$

$(0, -\frac{1}{2}) = (-\frac{1}{4}, 0)$

iii) $\frac{\partial f}{\partial x} = 0$ for $y \cdot (1-x+y) = x \cdot (1-x+2y)$

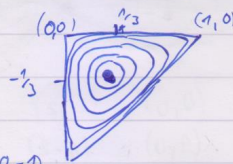
for $y=0$ or $y=2x-1$

$\frac{\partial f}{\partial y} = 0$ for $x=0$ or $x=2y+1$

within the triangle

$y = 4y + 2 - 1 \Rightarrow y = -\frac{1}{3} \Rightarrow x = \frac{1}{3}$ is $(\frac{1}{3}, -\frac{1}{3})$

$f(\frac{1}{3}, -\frac{1}{3}) = -\frac{1}{9} \cdot (\frac{1}{3}) = -\frac{1}{27}$ minimum



b) $g = y^2(1-x) - x^2(1-x)$

$\frac{\partial g}{\partial x} = -y^2 - 2x - 3x^2$

$\frac{\partial g}{\partial y} = 2y - 2yx$

$= 0$ for $y=0$ or $x=1$

$\frac{\partial g}{\partial x}(x, 0) = -2x - 3x^2 = -x(2+3x)$

$= 0$ for $x=0$ and $x=-\frac{2}{3}$

$\frac{\partial g}{\partial x}(1, 0) = -y^2 - 5 \neq 0$

$(0, 0) ; (0, -\frac{2}{3})$

$f_{xx} = -2 - 6x$

$f_{xy} = -2y$

$f_{yy} = 2 - 2x$

$(0, 0)$ is saddle

$(0, -\frac{2}{3})$ is a minimum

18S

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

19X*

No image has yet been uploaded for this question
No solution has yet been submitted for this question.

20Y*

No image has yet been uploaded for this question
No solution has yet been submitted for this question.