

2017 Mathematics (2)

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Section A

1

Consider the two intersecting lines given by equations

$$\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \quad \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

where s and t are real parameters.

(a) At what angle do the lines intersect? [1]

(b) Find the point at which they intersect. [1]

2

Consider $f(z) = ze^{iz}$, where $z = x + iy$ and x and y are real.

(a) Find the real part of $f(z)$. [1]

(b) Find the imaginary part of $f(z)$. [1]

3

Consider the matrix

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$

where $a \neq 0$ is a real number.

(a) Compute the matrix's eigenvalues. [1]

(b) Find its normalised eigenvectors. [1]

4

Find the first two non-zero terms in the Taylor series expansion of $x^3 \cos^2 x$ around the point $x = 0$. [2]

5

Find the first two non-zero terms in the Fourier series expansion of the function $\cos^4 x$, defined on $-\pi \leq x < \pi$. [2]

6

Consider the two vector fields

$$\mathbf{F} = (\sin x, \sin y, \sin z), \quad \mathbf{G} = (\cos x, \cos y, \cos z).$$

(a) Calculate $\mathbf{F} \times \mathbf{G}$. [1]

(b) Hence find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$. [1]

7

Consider the ordinary differential equation

$$\frac{d^2 y}{dx^2} + 9y = -7 \cos 4x.$$

(a) Calculate its complementary function. [1]

(b) Calculate its particular integral. [1]

8

If $\mathbf{F} = (y^2, x^2, 0)$, compute the surface integral

$$\int_S \mathbf{F} \cdot d\mathbf{S}$$

where

(a) S is a circular disk in the x - y plane centred on the origin with unit radius, and with surface normal pointing in the positive z -direction, [1]

(b) S is a square in the x - z plane centred on the origin with sides of unit length parallel to the x - and z -axes and with surface normal pointing in the positive y -direction. [1]

9

Consider the twice-differentiable function $u = u(\xi)$.

(a) If $\xi = x + 2\sqrt{y}$, calculate $\partial^2 u / \partial y^2$. [1]

(b) Show by substitution that $u(x + 2\sqrt{y})$ solves the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2} \frac{\partial u}{\partial y} - y \frac{\partial^2 u}{\partial y^2} = 0. \quad [1]$$

10

A finite population of cockatiels has equal numbers of males and females. The probability that a male can sing is p . The probability that a female can sing is q .

(a) What is the probability that a cockatiel randomly selected from the population can sing? [1]

(b) A cockatiel is observed to sing. What is the probability that it is male? [1]

Section B

11T

(a) For the three position vectors \mathbf{a} , \mathbf{b} and \mathbf{c} show explicitly using components ($\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$, etc.) that the vector triple product can be expressed as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}. \quad [5]$$

Hence, using properties of the scalar triple product (or otherwise), show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}). \quad [5]$$

[Note that $\mathbf{a} \times \mathbf{b} \equiv \mathbf{a} \wedge \mathbf{b}$.]

(b) Write down the equation for a sphere S given that its centre is at position vector \mathbf{a} and its radius is $p > 0$. [2]

Now suppose there is a second sphere S' with its centre at \mathbf{b} and radius $q > 0$. What conditions must \mathbf{a} , \mathbf{b} , p and q satisfy in order for the two spheres S and S' to intersect in a circle? [4]

If S and S' do intersect, show that the plane in which the circle of intersection lies is given by

$$2(\mathbf{b} - \mathbf{a}) \cdot \mathbf{r} = p^2 - q^2 + b^2 - a^2,$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. [4]

12X

(a) (i) Write down the infinitesimally small volume element in spherical polar coordinates: (r, θ, ϕ) . [2]

(ii) Assume that the Earth is a sphere of radius R and that the density ρ of the atmosphere varies with height h above the surface as $\rho = \rho_0 \exp(-h/h_0)$ where ρ_0 and h_0 are positive constants. Find an integral expression for the mass of the atmosphere and integrate to obtain an explicit formula in terms of ρ_0 , h_0 and R . [8]

(b) (i) Sketch the region of integration for the following double integral:

$$\int_{x=-a}^a \int_{y=x^2}^{y=\sqrt{1-x^2}} dy dx,$$

where $a^2 = (\sqrt{5} - 1)/2$. [4]

(ii) Evaluate the integral, giving your answer in the form $A \sin^{-1} a + Ba^3$ where A and B are to be found. [6]

13Z

If a vector field can be written as the gradient of some scalar field, $\mathbf{F} = \nabla\Phi$, the vector field is said to be ‘conservative’.

(a) Show, using Cartesian coordinates, that the line integral $\int_C \mathbf{F} \cdot d\mathbf{x}$ of a conservative vector field, \mathbf{F} , along some path, C , can be calculated by using the scalar field evaluated at the end points. [2]

(b) Show, using Cartesian coordinates, that the curl of a conservative vector field is everywhere zero. [3]

(c) Calculate the curl of the vector field

$$\mathbf{F} = (2xy - z^3, x^2 - 2y, -3xz^2 - 1),$$

and thereby show that \mathbf{F} is conservative. [3]

(d) Calculate the underlying scalar field Φ by evaluating the line integral of \mathbf{F} along the piecewise linear path joining $(0, 0, 0)$ to $(x, 0, 0)$ to $(x, y, 0)$ to (x, y, z) . Why is the result undefined with respect to an additive constant? [6]

(e) Calculate explicitly the line integral of \mathbf{F} along the parabolic path described by (t, t, t^2) from $t = 1$ to $t = 2$. [6]

14R

(a) Suppose X is a discrete random variable taking positive integer values $0, 1, 2, \dots$. Its probability distribution is denoted by $P(X)$. Write down expressions for the mean μ and variance σ^2 . [2]

(b) When Cambridge United football team play a game, the probability that the total number of goals scored is X is given by

$$P(X) = A \frac{\lambda^X}{X!}, \quad (\dagger)$$

where A is a normalisation constant and λ is a positive constant.

(i) If P is normalised, show that $A = e^{-\lambda}$. [2]

(ii) In any game Cambridge United plays, what is the probability, in terms of λ , that K goals or fewer are scored? [1]

(iii) Show that the mean of the distribution P is λ . [5]

(c) In one season, the Cambridge United team play 10 games of football. You may assume that the probability of goal-scoring in every game is given by equation (\dagger) .

(i) What is the probability that at least one goal is scored in every game of the season? [2]

(ii) Show that the probability that only 1 goal is scored in total during the team’s entire season is $10\lambda e^{-10\lambda}$. [3]

(iii) Calculate the probability that 2 goals are scored in total during the team’s season. [5]

15Y

(a) Find the relevant integrating factor and solve the following equations:

$$(i) \quad (2xy^2 - y)dx + (2x - x^2y)dy = 0, \quad [5]$$

$$(ii) \quad (2y \sin x + 3y^4 \sin x \cos x)dx - (4y^3 \cos^2 x + \cos x)dy = 0. \quad [5]$$

You may give these solutions in implicit form.

(b) Consider an equation of the form

$$y = p x + f(p),$$

where $p \equiv \frac{dy}{dx}$ and f is a differentiable function. Show that

$$[x + f'(p)] \frac{dp}{dx} = 0$$

where $f'(p) \equiv \frac{df}{dp}$. [2]

Hence, or otherwise, find all solutions for the equation

$$y = p x + \frac{1}{p-1}. \quad [8]$$

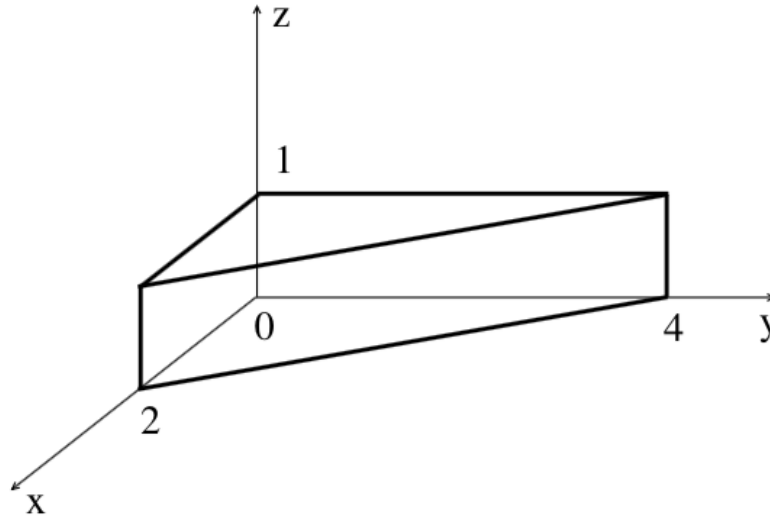
16X

(a) For the vector field $\mathbf{F}(x, y, z)$ give formulae in Cartesian coordinates for:

(i) $\nabla \cdot \mathbf{F}$, [1]

(ii) $\nabla \times \mathbf{F}$. [2]

(b) The closed surface S consists of the right triangular prism shown below.



For the vector field $\mathbf{F} = (0, (y + 2x - 4)^2, 1 - z^2)$:

(i) Calculate the outward flux for each of the five faces of the prism, and hence the total outward flux from S . [6]

(ii) Calculate $\nabla \cdot \mathbf{F}$. [3]

(iii) Find the volume integral of $\nabla \cdot \mathbf{F}$ over the interior of the prism. [6]

(iv) Comment on the relation between your answers to parts (b)(i) and (b)(iii). [2]

We can treat the following coupled system of differential equations as an eigenvalue problem:

$$\begin{aligned}2\frac{dy_1}{dt} &= 2f_1 - 3y_1 + y_2, \\2\frac{dy_2}{dt} &= 2f_2 + y_1 - 3y_2, \\ \frac{dy_3}{dt} &= f_3 - 4y_3,\end{aligned}$$

where f_1 , f_2 and f_3 is a set of time-dependent sources, and y_1 , y_2 and y_3 is a set of time-dependent responses.

- (a) If these equations are written using matrix notation,

$$\frac{d\mathbf{y}}{dt} + \mathbf{K}\mathbf{y} = \mathbf{f},$$

what are the elements of \mathbf{K} ? Find the eigenvalues and eigenvectors of \mathbf{K} . [6]

- (b) In the case when the system is not excited, $\mathbf{f} = \mathbf{0}$, find all of the solutions having the form

$$\mathbf{y}(t) = \mathbf{y}(0)e^{-\gamma t},$$

where $\gamma > 0$ is a decay constant. [4]

- (c) If \mathbf{f} is held constant at \mathbf{f}_0 , the response vector \mathbf{y} has the steady state value \mathbf{y}_0 (that is, with $\frac{d\mathbf{y}}{dt} = 0$). Write down \mathbf{y}_0 in terms of \mathbf{f}_0 , and find \mathbf{y}_0 in the case where $\mathbf{f}_0 = (1, 1, 1)^T$. [6]

- (d) Assume that \mathbf{y} starts in the steady state solution \mathbf{y}_0 given in (c) with $\mathbf{f}_0 = (1, 1, 1)^T$. Now suppose the source function abruptly falls to zero, $\mathbf{f}_0 = (0, 0, 0)^T$, so that the response vector \mathbf{y} moves away from \mathbf{y}_0 . Writing \mathbf{y} as a linear combination of the allowed solutions found in (b), derive an expression for the subsequent time evolution of the system. [4]

- (a) Suppose $f(x)$ is a 2π -periodic function defined on $-\pi \leq x < \pi$. Write down its Fourier series and give expressions for the coefficients appearing in it. Using the orthogonality relations or otherwise, determine the value of

$$\frac{1}{\pi} \int_{-\pi}^{\pi} (f(x))^2 dx$$

in terms of the Fourier coefficients of f (Parseval's identity). [7]

- (b) Show that the Fourier series of the 2π -periodic function $g(x) = x^3 - \pi^2 x$ for $-\pi \leq x < \pi$ is given by

$$g(x) = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \sin nx,$$

where the integer p should be determined. [7]

- (c) Using Parseval's identity for g , show that

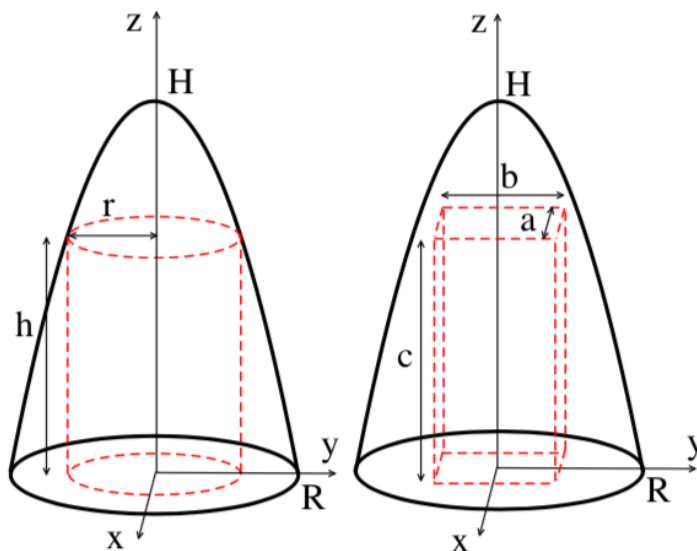
$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

[6]

The interior region of a paraboloid of height H and radius R of the base is defined by the following inequalities

$$0 < z < H \left[1 - (x^2 + y^2)/R^2 \right] .$$

Either a cylinder of height h and radius r or a rectangular parallelepiped with sides a , b and c can be inscribed into the paraboloid as shown by dashed lines in the left and right panels of the diagram, respectively.



By using the method of Lagrange multipliers,

- (a) show that the maximum possible volume of a cylinder, V_c , inscribed into the paraboloid as shown in the diagram above is

$$V_c = \frac{\pi R^2 H}{4} , \quad [7]$$

- (b) find in terms of H and R the maximum possible volume of the rectangular parallelepiped, V_p , inscribed into the paraboloid, [11]
(c) and thus determine which shape can produce a larger volume. [2]

[Hint: You need not prove that suitable extrema you find are actually maxima.]

(a) (i) Solve the equation

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{1+x},$$

subject to the boundary condition $y(0) = 1$. [4]

(ii) Solve the equation

$$\frac{dy}{dx} + \frac{1}{3}y = e^x y^4,$$

subject to the boundary condition $y(0) = 1$. [5]

(b) The following partial differential equation on the given interval,

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \quad 0 \leq x \leq L, \quad t \geq 0, \quad (\ddagger)$$

has the boundary conditions $u(0, t) = u(L, t) = 0$. By using the separable function $u(x, t) = X(x)T(t)$, show that the equation (\ddagger) may be written as

$$\frac{1}{T} \frac{dT}{dt} + 1 = \frac{1}{X} \frac{d^2 X}{dx^2} = -k^2,$$

with k a constant.

Determine the functions $X(x), T(t)$ satisfying the boundary conditions.

Hence, write down the general solution of the partial differential equation (\ddagger) . [11]