2017 Mathematics (2)

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Section A

1

Consider the two intersecting lines given by equations

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \qquad \boldsymbol{r} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix},$$

where s and t are real parameters.

(a) At what angle do the lines intersect? [1]

(b) Find the point at which they intersect. [1]

Solution(s):

From user: lester

(a) $\underline{r}_1 = \underline{a} + s\underline{b}$ angle of intersection $\theta = \cos^{-1}(\hat{\underline{b}} . \hat{\underline{d}}) = \cos^{-1}\left(\frac{0}{\text{something}}\right) = \frac{\pi}{2}$.

$\underline{r}_2 = \underline{c} + t\underline{d}$

(b) Intersect when $\underline{a} + s\underline{b} = \underline{c} + t\underline{d}$

$\Rightarrow \underline{a}.\underline{b} + s|\underline{b}|^2 = \underline{c}.\underline{b}$ (since $\underline{b}.\underline{d} = 0$)

$\Rightarrow s = \frac{\underline{c}.\underline{b} - \underline{a}.\underline{b}}{|\underline{b}|^2} = \frac{(\underline{c} - \underline{a}).\underline{b}}{|\underline{b}|^2}$

$\Rightarrow$ intersect at $\underline{r} = \underline{a} + (\underline{c} - \underline{a}).\hat{\underline{b}}\,\hat{\underline{b}} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} . \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + 0 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$.

2

Consider $f(z) = z\mathrm{e}^{\mathrm{i}z}$, where $z = x + \mathrm{i}y$ and x and y are real.

(a) Find the real part of $f(z)$. [1]

(b) Find the imaginary part of $f(z)$. [1]

Solution(s):

From user: lester

$$f(z) = ze^{iz} \Rightarrow f = (x+iy)e^{-y}e^{ix} = (x+iy)e^{-y}(\cos x + i\sin x)$$

$$\therefore \mathrm{Re}(f) = e^{-y}(x\cos x - y\sin x),$$

$$\mathrm{Im}(f) = e^{-y}(y\cos x + x\sin x).$$

3

Consider the matrix

$$\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix},$$

where $a \neq 0$ is a real number.

(a) Compute the matrix's eigenvalues. [1]

(b) Find its normalised eigenvectors. [1]

Solution(s):

From user: lester

$$M = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}, \quad a \neq 0. \quad M\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} \quad \& \quad M\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -a \\ a \end{pmatrix}$$

$\therefore$ M has evals $+a$ & $-a$ with normalised evecs $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ respectively.

4

Find the first two non-zero terms in the Taylor series expansion of $x^3 \cos^2 x$ around the point $x = 0$. [2]

Solution(s):

From user: lester

$$x^3\cos^2 x = x^3\left(1 - \frac{x^2}{2!} + \ldots\right)^2 = x^3\left(1 - x^2 + \ldots\right) = x^3 - x^5 + \ldots.$$

5

Find the first two non-zero terms in the Fourier series expansion of the function $\cos^4 x$, defined on $-\pi \leqslant x < \pi$. [2]

Solution(s):

From user: lester

$$\boxed{\cos^2 x = \tfrac{1}{2}(\cos 2x + 1)}$$

$$\cos^4 x = (\cos^2 x)^2 = \left(\tfrac{1}{2}(\cos 2x + 1)\right)^2 = \tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cos^2 2x$$

$$= \tfrac{1}{4} + \tfrac{1}{2}\cos 2x + \tfrac{1}{4}\cdot\tfrac{1}{2}(\cos 4x + 1) = \underbrace{\tfrac{3}{8} + \tfrac{1}{2}\cos 2x}_{\text{first two terms}} + \tfrac{1}{8}\cos 4x.$$

6

Consider the two vector fields

$$\boldsymbol{F} = (\sin x, \sin y, \sin z), \qquad \boldsymbol{G} = (\cos x, \cos y, \cos z).$$

(a) Calculate $\boldsymbol{F} \times \boldsymbol{G}$. [1]

(b) Hence find $\nabla \cdot (\boldsymbol{F} \times \boldsymbol{G})$. [1]

Solution(s):

From user: lester

$$\underline{F} = (s_x, s_y, s_z) \qquad s = \text{"sin"}$$
$$\underline{G} = (c_x, c_y, c_z) \qquad c = \text{"cos"}$$

(a) $$\underline{F} \wedge \underline{G} = \begin{pmatrix} s_y c_z - s_z c_y \\ s_z c_x - s_x c_z \\ s_x c_y - s_y c_x \end{pmatrix}$$

(b) $\underline{\nabla} \cdot (\underline{F} \wedge \underline{G}) = 0$ as no x in $(\underline{F} \wedge \underline{G})_x$, etc.

7

Consider the ordinary differential equation

$$\frac{d^2y}{dx^2} + 9y = -7\cos 4x.$$

(a) Calculate its complementary function. [1]

(b) Calculate its particular integral. [1]

Solution(s):

From user: lester

$$\frac{d^2y}{dx^2} + 9y = -7\cos 4x .$$

(a) $y_{CF} = A\cos 3x + B\sin 3x$ (by inspection).

(b) $y_{PI} = \lambda\cos 4x \; ; \quad -16\lambda + 9\lambda = -7 \quad \Rightarrow \lambda = 1 \quad \Rightarrow \quad y_{PI} = \cos 4x .$

8

If $\boldsymbol{F} = (y^2, x^2, 0)$, compute the surface integral

$$\int_S \boldsymbol{F}\cdot d\boldsymbol{S}$$

where

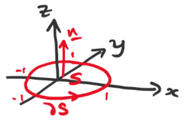
(a) S is a circular disk in the x-y plane centred on the origin with unit radius, and with surface normal pointing in the positive z-direction, [1]

(b) S is a square in the x-z plane centred on the origin with sides of unit length parallel to the x- and z-axes and with surface normal pointing in the positive y-direction. [1]

Solution(s):

From user: lester

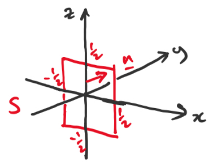
$\underline{F} = \begin{pmatrix} y^2 \\ x^2 \\ 0 \end{pmatrix}$



$\underline{\nabla}\wedge\underline{F} = \begin{pmatrix} 0 \\ 0 \\ 2x-2y \end{pmatrix}$

(a) $\int_S \underline{F}\cdot\underline{dS} = \int_{\text{stuff}} \begin{pmatrix} ? \\ ? \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} d? = 0 .$

(b)



$\int_S \underline{F}\cdot\underline{dS} = \int_{x=-\frac{1}{2}}^{\frac{1}{2}} \int_{z=-\frac{1}{2}}^{\frac{1}{2}} \begin{pmatrix} 0 \\ x^2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dz\,dx = \left[\tfrac{1}{3}x^3\right]_{-\frac{1}{2}}^{\frac{1}{2}} \left[z\right]_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{4}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) = \frac{1}{12} .$

9

Consider the twice-differentiable function $u = u(\xi)$.

(a) If $\xi = x + 2\sqrt{y}$, calculate $\partial^2 u/\partial y^2$. [1]

(b) Show by substitution that $u(x + 2\sqrt{y})$ solves the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u}{\partial y} - y\frac{\partial^2 u}{\partial y^2} = 0\,.$$ [1]

Solution(s):

From user: lester

$$\xi = x + 2\sqrt{y}\,, \quad u = u(\xi)$$

(a) $$\frac{\partial u}{\partial y} = \frac{du}{d\xi}\frac{\partial \xi}{\partial y} = u' \cdot \frac{1}{\sqrt{y}}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{1}{2}u' y^{-\frac{3}{2}} + \frac{1}{\sqrt{y}} u'' \frac{1}{\sqrt{y}} = \frac{u''}{y} - \frac{1}{2}\frac{u'}{y^{3/2}}.$$

(b) Similarly $\frac{\partial u}{\partial x} = u'$ & $\frac{\partial^2 u}{\partial x^2} = u''$.

$$\therefore \frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\frac{\partial u}{\partial y} - y\frac{\partial^2 u}{\partial y^2} = u'' - \frac{1}{2}\frac{u'}{\sqrt{y}} - y\left(\frac{u''}{y} - \frac{1}{2}\frac{u'}{y^{3/2}}\right) = 0. \quad \text{QED.}$$

10

A finite population of cockatiels has equal numbers of males and females. The probability that a male can sing is p. The probability that a female can sing is q.

(a) What is the probability that a cockatiel randomly selected from the population can sing? [1]

(b) A cockatiel is observed to sing. What is the probability that it is male? [1]

Solution(s):

From user: lester

(a) $$P(\text{singer}) = \tfrac{1}{2}p + \tfrac{1}{2}q$$

(b) $$P(\text{male}\,|\,\text{singer}) = \frac{p(\text{singer}\,|\,\text{male})\,p(\text{male})}{p(\text{singer})} = \frac{p \cdot \frac{1}{2}}{\frac{1}{2}(p+q)} = \frac{p}{p+q}.$$

Section B

11T

(a) For the three position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ show explicitly using components ($\mathbf{a} = a_x\hat{\mathbf{i}} + a_y\hat{\mathbf{j}} + a_z\hat{\mathbf{k}}$, etc.) that the vector triple product can be expressed as

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,. \qquad [5]$$

Hence, using properties of the scalar triple product (or otherwise), show that

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\,. \qquad [5]$$

[*Note that* $\mathbf{a} \times \mathbf{b} \;\equiv\; \mathbf{a} \wedge \mathbf{b}$.]

(b) Write down the equation for a sphere S given that its centre is at position vector $\mathbf{a}$ and its radius is $p > 0$. [2]

Now suppose there is a second sphere S' with its centre at $\mathbf{b}$ and radius $q > 0$. What conditions must $\mathbf{a}$, $\mathbf{b}$, p and q satisfy in order for the two spheres S and S' to intersect in a circle? [4]

If S and S' do intersect, show that the plane in which the circle of intersection lies is given by

$$2(\mathbf{b} - \mathbf{a}) \cdot \mathbf{r} = p^2 - q^2 + b^2 - a^2\,,$$

where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$. [4]

Solution(s):

From user: lester

(a) I don't like the inflexible way this question requires we work in components. ☹

$$\underline{b} \wedge \underline{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{pmatrix} b_2c_3 - c_2b_3 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$$

$$\therefore \underline{a} \wedge (\underline{b} \wedge \underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_2c_3 - c_2b_3 & b_3c_1 - b_1c_3 & b_1c_2 - b_2c_1 \end{vmatrix} = \begin{pmatrix} a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_2 \\ \text{and another line} \\ \text{and a third line} \end{pmatrix}$$

$$= \begin{pmatrix} b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3) \\ \text{and another line} \\ \text{and a third line} \end{pmatrix} = \begin{pmatrix} b_1\, \underline{a}\cdot\underline{c} - c_1\, \underline{a}\cdot\underline{b} \\ b_2\, \underline{a}\cdot\underline{c} - c_2\, \underline{a}\cdot\underline{b} \\ b_3\, \underline{a}\cdot\underline{c} - c_3\, \underline{a}\cdot\underline{b} \end{pmatrix} = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$$

QED.

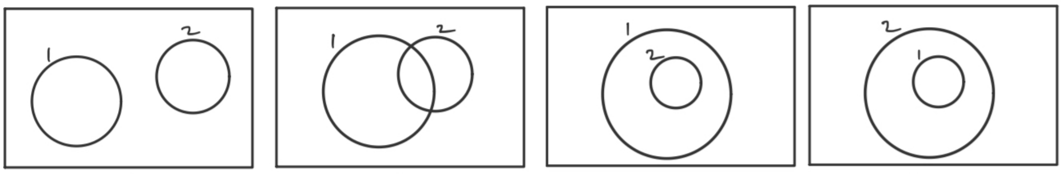
! HORRIBLE !

$$\underline{a} \wedge \underline{b} \cdot \underline{c} \wedge \underline{d} = \varepsilon_{ijk} a_j b_k \varepsilon_{ilm} c_l d_m = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) a_j b_k c_l d_m$$

$$= a_l b_m c_l d_m - a_m b_l c_l d_m = (\underline{a}\cdot\underline{c})(\underline{b}\cdot\underline{d}) - (\underline{a}\cdot\underline{d})(\underline{b}\cdot\underline{c}). \quad \text{QED}$$

(b) Sphere 1: $|\underline{r} - \underline{a}| = p$. Sphere 2: $|\underline{r} - \underline{b}| = q$.

Possible sphere configurations are:



A B C D

A happens iff $|\underline{a} - \underline{b}| > p + q$.

C happens iff $|\underline{b} - \underline{a}| + q < p$.

D happens iff $|\underline{b} - \underline{a}| + p < q$.

$\Rightarrow$ B happens iff $\left\{ \begin{array}{l} |\underline{a} - \underline{b}| \leq p + q \ \& \\ |\underline{a} - \underline{b}| + q \geq p \ \& \\ |\underline{a} - \underline{b}| + p \geq q \end{array} \right\}$

Intersect in a circle

Note that we include within "intersect in a circle" the degenerate "point contact" case. If this is undesired, $\leq$ & $\geq$ can be replaced with $<$ & $>$.

When they intersect, points on the intersection satisfy:

$$\left\{ \begin{array}{l} |\underline{r} - \underline{a}| = p \\ |\underline{r} - \underline{b}| = q \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} \underline{r}^2 - 2\underline{r}\cdot\underline{a} + \underline{a}^2 = p^2 \\ \underline{r}^2 - 2\underline{r}\cdot\underline{b} + \underline{b}^2 = q^2 \end{array} \right\} \Rightarrow \left\{ 2(\underline{b} - \underline{a})\cdot\underline{r} + \underline{a}^2 - \underline{b}^2 = p^2 - q^2 \right\}.$$

Q.E.D.

12X

(a) (i) Write down the infinitesimally small volume element in spherical polar coordinates: (r, θ, ϕ). [2]

(ii) Assume that the Earth is a sphere of radius R and that the density ρ of the atmosphere varies with height h above the surface as $\rho = \rho_0 \exp(-h/h_0)$ where ρ_0 and h_0 are positive constants. Find an integral expression for the mass of the atmosphere and integrate to obtain an explicit formula in terms of ρ_0, h_0 and R. [8]

(b) (i) Sketch the region of integration for the following double integral:

$$\int_{x=-a}^{a} \int_{y=x^2}^{y=\sqrt{1-x^2}} dy\, dx\,,$$

where $a^2 = (\sqrt{5}-1)/2$. [4]

(ii) Evaluate the integral, giving your answer in the form $A \sin^{-1} a + Ba^3$ where A and B are to be found. [6]

Solution(s):

From user: cgl20

(a) (i) $dV = r^2 \sin\theta\, dr\, d\theta\, d\phi$

(ii) $\rho = \rho_0 e^{-h/h_0}$

$\therefore M = \int_{r=R}^{\infty} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \rho_0 e^{-\frac{r-R}{h_0}} r^2 \sin\theta\, dr\, d\theta\, d\phi$

$= 4\pi\rho_0 e^{R/h_0} \underbrace{\int_{r=R}^{\infty} r^2 e^{-r/h_0}\, dr}$ call this I_2: $I_n = \int_{r=R}^{\infty} r^n e^{-r/h_0}\, dr = \left[r^n h_0 e^{-r/h_0}\right]_{\infty}^{R} + h_0 n \int_R^{\infty} r^{n-1} e^{-r/h_0}\, dr$

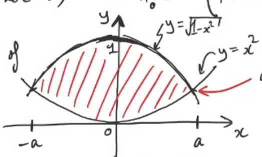
$= R^n h_0 e^{-R/h_0} + n h_0 I_{n-1}$

$\Rightarrow I_2 = R^2 h_0 e^{-R/h_0} + 2h_0 \left(R h_0 e^{-R/h_0} + h_0 \left(h_0 e^{-R/h_0}\right)\right)$

$= h_0 e^{-R/h_0} \left(R^2 + 2Rh_0 + 2h_0^2\right)$

$= 4\pi\rho_0 h_0 (R^2 + 2Rh_0 + 2h_0^2)$.
(from working over here $\rightarrow$)

(b) (i) $I = \int_{x=-a}^{a} \int_{y=x^2}^{\sqrt{1-x^2}} dy\, dx$ is the area of



at intersection, $\sqrt{1-x^2} = x^2 \leftarrow \circledast$

$\Rightarrow 1 - x^2 = x^4$

$\Rightarrow x^2 = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} = a^2$

actually, only "+" solution is allowed if LHS of $\circledast$ is to stay real

(ii) $\therefore I = \int_{x=-a}^{a} [y]_{x^2}^{\sqrt{1-x^2}}\, dx = \int_{-a}^{a} \sqrt{1-x^2} - x^2\, dx = -\frac{2}{3}a^3 + \underbrace{\int_{-a}^{a} \sqrt{1-x^2}\, dx}_{= J}$. Let $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$J = \int_{-\sin^{-1}(a)}^{\sin^{-1}(a)} \cos^2\theta\, d\theta = \int_{-\sin^{-1}(a)}^{+\sin^{-1}(a)} \frac{1}{2}(\cos 2\theta + 1)\, d\theta$

$= \left[\frac{1}{4}\sin 2\theta + \frac{\theta}{2}\right]_{-\sin^{-1}(a)}^{\sin^{-1}(a)} = \frac{1}{2}\sin\left(2\sin^{-1}(a)\right) + \sin^{-1}(a)$

$= a\cos\left(\sin^{-1} a\right) + \sin^{-1} a$

$\sin\theta = a$
$\theta = \sin^{-1} a$
$\cos\theta = \sqrt{1-a^2}$

$= a\sqrt{1-a^2} + \sin^{-1} a$

$\therefore \underline{I = a\sqrt{1-a^2} + \sin^{-1} a - \frac{2}{3}a^3}$

$1 - a^2 = 1 - \frac{\sqrt{5}-1}{2} = \frac{3}{2} - \frac{\sqrt{5}}{2}$

But question requests form built from a^3 terms:

$a\sqrt{1-a^2} = a^3\sqrt{\frac{1}{a^4} - \frac{1}{a^2}} = a^3\sqrt{\frac{4}{(\sqrt{5}-1)^2} - \frac{2}{\sqrt{5}-1}} = a^3\sqrt{\frac{4 - 2\sqrt{5} + 2}{(\sqrt{5}-1)^2}} = a^3\sqrt{\frac{6-2\sqrt{5}}{6-2\sqrt{5}}} = a^3$, so:

$I = a^3 + \sin^{-1} a - \frac{2}{3}a^3 = \frac{1}{3}a^3 + \sin^{-1} a$. In the question's notation: $A = 1, B = \frac{1}{3}$.

If a vector field can be written as the gradient of some scalar field, $\boldsymbol{F} = \nabla\Phi$, the vector field is said to be 'conservative'.

(a) Show, using Cartesian coordinates, that the line integral $\int_{\mathcal{C}} \boldsymbol{F} \cdot \boldsymbol{d}\mathbf{x}$ of a conservative vector field, $\boldsymbol{F}$, along some path, $\mathcal{C}$, can be calculated by using the scalar field evaluated at the end points. [2]

(b) Show, using Cartesian coordinates, that the curl of a conservative vector field is everywhere zero. [3]

(c) Calculate the curl of the vector field

$$\boldsymbol{F} = \left(2xy - z^3,\ x^2 - 2y,\ -3xz^2 - 1\right),$$

and thereby show that $\boldsymbol{F}$ is conservative. [3]

(d) Calculate the underlying scalar field Φ by evaluating the line integral of $\boldsymbol{F}$ along the piecewise linear path joining $(0,0,0)$ to $(x,0,0)$ to $(x,y,0)$ to (x,y,z). Why is the result undefined with respect to an additive constant? [6]

(e) Calculate explicitly the line integral of $\boldsymbol{F}$ along the parabolic path described by (t,t,t^2) from $t=1$ to $t=2$. [6]

Solution(s):

From user: lester

$\underline{F} = \nabla\Phi$

(a) $\int_C \underline{F}\cdot d\underline{x} = \int_C \nabla\Phi\cdot d\underline{x} = \int_C\left(\frac{\partial\Phi}{\partial x}dx + \frac{\partial \Phi}{\partial y}dy + \dots\right) = \int_C d\Phi = \left[\Phi\right]_C$ QED.

(b) $(\nabla\wedge\nabla\Phi)_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k}\Phi$. $\varepsilon_{ijk} = -\varepsilon_{ikj}$ but $\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_k} = \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j}$ so $(\nabla\wedge\nabla\Phi)_i = 0 \ \forall i$.

(c) $\underline{F} = (2xy - z^3,\ x^2-2y,\ -3xz^2-1) = \nabla\big(\underbrace{x^2y - xz^3 - y^2 - z}_{\Phi \text{ up to a constant.}}\big)$ so $\underline{F}$ is conservative & $\nabla\wedge\underline{F} = \underline{0}$

(d) We already found Φ in (c), up to a const. Nonetheless, we are asked to find it another more laborious way:

Let $\Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3$ with $\left\{\begin{array}{ll} \Gamma_1 = (tx, 0, 0), & d\underline{x} = (x,0,0)\,dt \\ \Gamma_2 = (x, ty, 0), & d\underline{x} = (0,y,0)\,dt \\ \Gamma_3 = (x, y, tz), & d\underline{x} = (0,0,z)\,dt \end{array}\right\}$.

Then $\Phi(\underline{x}) - \Phi(0) = \int_\Gamma \underline{F}\cdot d\underline{x} = \int_{t=0}^{1} dt\Big(\underbrace{\big(2(tx)0 - 0^3\big)x}_{\Gamma_1 \text{ part}} + \underbrace{\big(x^2 - 2(ty)\big)y}_{\Gamma_2 \text{ part}} + \underbrace{\big(-3x(tz)^2 - 1\big)z}_{\Gamma_3 \text{ part}}\Big)$

$= \int_{t=0}^{1} dt\Big((x^2y - z) - (2y^2)t - (3xz^2)t^2\Big) = \Big[(x^2y - z)t - y^2t^2 - xz^2t^3\Big]_0^1 = x^2y - z - y^2 - xz^2$.

This is also only Φ up to a constant as it is $\Phi(\underline{x}) - \Phi(0)$.

(e) With $\Gamma = (t, t, t^2)$ then $d\underline{x} = (1, 1, 2t)\,dt$ and

$\int_{t=1}^{2} \underline{F}\cdot d\underline{x} = \int_1^2 dt\Big((2t^2 - t^6).1 + (t^2 - 2t).1 + (-3t^5 - 1)2t\Big)$

$= \int_1^2 dt\left(-4t + 3t^2 - 7t^6\right) = \left[-2t^2 + t^3 - t^7\right]_1^2 = (\cancel{-8}\cancel{+8} - 2^7) - (-2 \cancel{+1} \cancel{-1}) = 2 - 128 = -126$.

Check: $\Phi(2,2,4) - \Phi(1,1,1) = (\cancel{8} - 2.4^3 \cancel{-4} \cancel{-4}) - (\cancel{1} \cancel{-1} - 1 - 1) = -2^7 + 2 = \boxed{-126}$ ✓

14R

(a) Suppose X is a discrete random variable taking positive integer values 0, 1, 2, ... Its probability distribution is denoted by $P(X)$. Write down expressions for the mean μ and variance σ^2. [2]

(b) When Cambridge United football team play a game, the probability that the total number of goals scored is X is given by

$$P(X) = A\frac{\lambda^X}{X!}, \qquad (\dagger)$$

where A is a normalisation constant and λ is a positive constant.

(i) If P is normalised, show that $A = e^{-\lambda}$. [2]

(ii) In any game Cambridge United plays, what is the probability, in terms of λ, that K goals or fewer are scored? [1]

(iii) Show that the mean of the distribution P is λ. [5]

(c) In one season, the Cambridge United team play 10 games of football. You may assume that the probability of goal-scoring in every game is given by equation ($\dagger$).

(i) What is the probability that at least one goal is scored in every game of the season? [2]

(ii) Show that the probability that only 1 goal is scored in total during the team's entire season is $10\lambda e^{-10\lambda}$. [3]

(iii) Calculate the probability that 2 goals are scored in total during the team's season. [5]

Solution(s):

From user: lester

(a) This question says "positive integer values" but then lists non-negative integers 0, 1, 2, 3, ... Presumably it means the latter. Poor proof reading!

$$\mu = \sum_{x=0}^{\infty} x\,P(X=x), \quad \sigma^2 = \sum_{x=0}^{\infty}(x-\mu)^2 P(X=x).$$

(b) (i) $\underline{1} = \sum_{x=0}^{\infty} P(X=x) = \sum_{x=0}^{\infty} A\frac{\lambda^x}{x!} = A\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = Ae^{\lambda} \therefore A = e^{-\lambda}$. QED

(ii) $P(X \leq K) = \sum_{x=0}^{K} P(X=x) = e^{-\lambda}\sum_{x=0}^{K}\frac{\lambda^x}{x!}$.

(iii) $\mu = \sum_{x=0}^{\infty} x\left(e^{-\lambda}\frac{\lambda^x}{x!}\right) = \sum_{x=1}^{\infty} e^{-\lambda}\frac{\lambda^{x-1}\lambda}{(x-1)!} = \lambda e^{-\lambda}\sum_{t=0}^{\infty}\frac{\lambda^t}{t!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$. QED

(c) (i) $P(\text{at least 1 goal in each of 10 games}) = (P(X \geq 1))^{10} = (1-P(X=0))^{10} = \left(1-\frac{\lambda^0 e^{-\lambda}}{0!}\right)^{10} = (1-e^{-\lambda})^{10}$.

(ii) $P(1 \text{ goal in } \boxed{\text{season}}) = \binom{10}{1}P(1,0,0,0,0,0,0,0,0,0) = 10 \cdot P(X=1)\,P(X=0)^9 = 10\left(\frac{\lambda^1}{1!}e^{-\lambda}\right)\left(\frac{\lambda^0}{0!}e^{-\lambda}\right)^9 = 10\lambda e^{-10\lambda}$ QED

10 games

(iii) $P(2 \text{ goals in season}) = \binom{10}{2}(P(X=1))^2(P(X=0))^8 + \binom{10}{1}P(X=2)(P(X=0))^9$

$= \frac{10\times 9}{2}\left(\frac{\lambda^1}{1!}e^{-\lambda}\right)^2\left(\frac{\lambda^0}{0!}e^{-\lambda}\right)^8 + 10\left(\frac{\lambda^2}{2!}e^{-\lambda}\right)\left(\frac{\lambda^0}{0!}e^{-\lambda}\right)^9$

$= e^{-10\lambda}\left(45\lambda^2 + 5\lambda^2\right) = 50\lambda^2 e^{-10\lambda}$.

(a) Find the relevant integrating factor and solve the following equations:

$$(i) \qquad (2xy^2 - y)dx + (2x - x^2y)dy = 0\,, \qquad [5]$$

$$(ii) \qquad (2y\sin x + 3y^4 \sin x \cos x)dx - (4y^3\cos^2 x + \cos x)dy = 0\,. \qquad [5]$$

You may give these solutions in implicit form.

(b) Consider an equation of the form

$$y = px + f(p)\,,$$

where $p \equiv \frac{dy}{dx}$ and f is a differentiable function. Show that

$$[x + f'(p)]\,\frac{dp}{dx} = 0$$

where $f'(p) \equiv \frac{df}{dp}$. [2]

Hence, or otherwise, find all solutions for the equation

$$y \;=\; px \;+\; \frac{1}{p-1}\,. \qquad [8]$$

Solution(s):

From user: lester

(a)(i) $\underbrace{(2xy^2-y)}_{P}dx + \underbrace{(2x-x^2y)}_{Q}dy = 0 \quad (*)$ $\quad \frac{\partial P}{\partial y} = 4xy-1,\ \frac{\partial Q}{\partial x} = 2-2xy$ so LHS$(*)$ is inexact.

$\frac{\partial}{\partial y}\left(\mu(2xy^2-y)\right) = \mu(4xy-1) + (2xy^2-y)\frac{\partial \mu}{\partial y}$ $\qquad$ Let $\mu = \mu(y)$

$\frac{\partial}{\partial x}\left(\mu(2x-x^2y)\right) = \mu(2-2xy) + (2x-x^2y)\frac{\partial \mu}{\partial x}$ $\qquad$ If μ is the I.F. then:

$\mu(4xy-1) + y(2xy-1)\frac{d\mu}{dy} = \mu(2-2xy)$

$\Rightarrow y(2xy-1)\frac{d\mu}{dy} = \mu(3-6xy)$

$= 3\mu(1-2xy)$

$\Rightarrow y\frac{d\mu}{dy} = -3\mu$ (or $xy = \frac{1}{2}$).

$\Rightarrow \ln\mu = -3\ln y \Rightarrow \mu = y^{-3}$ will work.

So it suffices to solve:

$\left(\frac{2x}{y} - \frac{1}{y^2}\right)dx + \left(\frac{2x}{y^3} - \frac{x^2}{y^2}\right)dy = 0$

$\Rightarrow d\left(\frac{x^2}{y} - \frac{x}{y^2}\right) = 0$

$\Rightarrow \frac{x^2}{y} - \frac{x}{y^2} = \text{const}$

(ii) By inspection:

$d(-y^4\cos^3 x - y\cos^2 x)$

$= (3y^4\cos^2 x\sin x + 2y\sin x\cos x)dx$

$+ (-4y^3\cos^3 x - \cos^2 x)dy$

$= (\text{LHS given})\cos x \quad \therefore \mu = \cos x$ works

$\therefore (2y\sin x + 3y^4\sin x\cos x)dx - (4y^3\cos^2 x + \cos x)dy = 0$

has soln $-y^4\cos^3 x - y\cos^2 x = -\text{const}$

$\Rightarrow y\cos^2 x(1 + y^3\cos x) = \text{const}$

(b) $y = px + f(p) \qquad p = \frac{dy}{dx}$

$(x + f'(p))\frac{dp}{dx} = \left(x + \frac{d}{dp}(y - px)\right)\frac{dp}{dx} = \left(\cancel{x} + \frac{dy}{dp} - p\frac{dx}{dp} - \cancel{x}\right)\frac{dp}{dx} = \frac{dy}{dx} - p = p - p = 0.$ QED (†)

$y = px + \frac{1}{p-1} \Rightarrow \cancel{px} + f(p) = \cancel{px} + \frac{1}{p-1} \Rightarrow f'(p) = \frac{-1}{(p-1)^2}$ $\qquad$ But (†) says $\frac{dp}{dx} = 0$ or $f'(p) = -x$

If $\frac{dp}{dx} = 0$ then $\frac{d^2y}{dx^2} = 0 \Rightarrow y = Ax + B$.

If $f'(p) = -x$ then $-x = \frac{-1}{(p-1)^2} \Rightarrow (p-1)^2 = \frac{1}{x} \Rightarrow \frac{dy}{dx} - 1 = \pm x^{-\frac{1}{2}} \Rightarrow y = x \pm 2x^{\frac{1}{2}} + c$

Check: $\left.\begin{array}{l} y = Ax + B \\ \Rightarrow p = A \end{array}\right\} \Rightarrow px + \frac{1}{p-1} = Ax + \frac{1}{A-1} \overset{\text{is supposed to equal}}{=} Ax + B \quad \therefore$ need $B = \frac{1}{A-1} \Rightarrow y = Ax + \frac{1}{A-1}$.

Check: $y = x \pm 2x^{\frac{1}{2}} + c \Rightarrow p = 1 \pm x^{-\frac{1}{2}} \Rightarrow px + \frac{1}{p-1} = x \pm x^{\frac{1}{2}} + \frac{1}{\pm x^{-\frac{1}{2}}} = x \pm 2x^{\frac{1}{2}} = y$ only if $c = 0$.

$\therefore$ Answer: $y = Ax + \frac{1}{A-1}$ for $A \neq 1$, or $y = x \pm 2\sqrt{x}$.

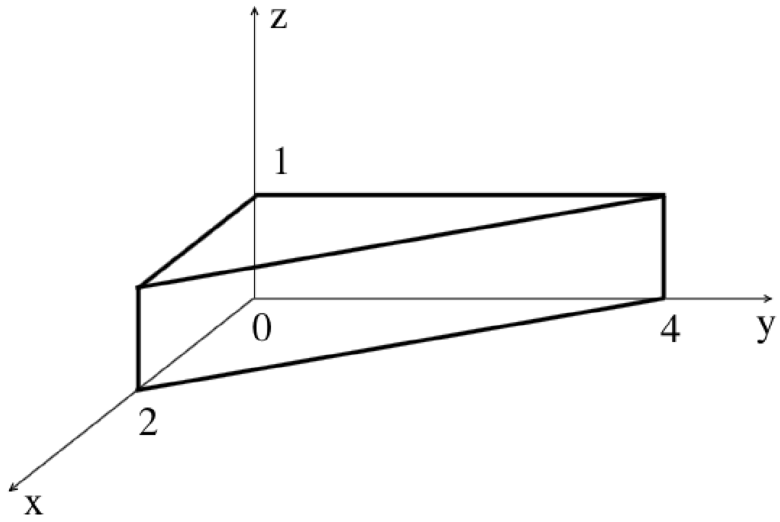
16X

(a) For the vector field $\boldsymbol{F}(x, y, z)$ give formulae in Cartesian coordinates for:

(i) $\nabla \cdot \boldsymbol{F}$, [1]

(ii) $\nabla \times \boldsymbol{F}$. [2]

(b) The closed surface S consists of the right triangular prism shown below.



For the vector field $\boldsymbol{F} = \left(0,\, (y + 2x - 4)^2,\, 1 - z^2\right)$:

(i) Calculate the outward flux for each of the five faces of the prism, and hence the total outward flux from S. [6]

(ii) Calculate $\nabla \cdot \boldsymbol{F}$. [3]

(iii) Find the volume integral of $\nabla \cdot \boldsymbol{F}$ over the interior of the prism. [6]

(iv) Comment on the relation between your answers to parts (b)(i) and (b)(iii). [2]

Solution(s):

From user: lester

(a) (i) $\underline{\nabla} \cdot \underline{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

(ii) $\left(\underline{\nabla} \wedge \underline{F}\right)_x = \dfrac{\partial F_z}{\partial y} - \dfrac{\partial F_y}{\partial z}$ and similarly for the other cpts $x \to y \to z \to x$.

(b) $\underline{F} = \left(0, (y+2x-4)^2, 1-z^2\right)$

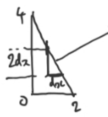
(i) $$\text{Flux}_{\text{bot}} = \int_{x=0}^{2}\int_{y=0}^{4-2x} (1-0^2).(-1)\, dy\, dx = -(\text{Area base}) = -4$$

$$\text{Flux}_{\text{top}} = \text{DITTO } (1-1^2).(+1)\, dy\, dx = 0$$

$$\text{Flux}_{xz} = \int_{x=0}^{2}\int_{z=0}^{1} (0+2x-4)^2.(-1)\, dz\, dx = \int_{x=2}^{0} (2x-4)^2\, dx = \left[\frac{1}{6}(2x-4)^3\right]_2^0 = \frac{1}{6}\left(-4^3\right) = -\frac{2^6}{2.3} = -\frac{32}{3}.$$

$$\text{Flux}_{yz} = \int_{y=0}^{4}\int_{z=0}^{1} 0.(-1)\, dy\, dz = 0$$

$$\text{Flux}_{\text{slanted}} = \int_{x=0}^{2}\int_{z=0}^{1} \begin{pmatrix} 0 \\ ((\cancel{4}-\cancel{2x})+\cancel{2x}-\cancel{4})^2 \\ 1-z^2 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} dz\, dx}_{d\underline{A}} = 0.$$



$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ direction $\therefore \underline{\hat{n}} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

$dA = dz\sqrt{dx^2 + (2dx)^2} = \sqrt{5}\, dz\, dx$

$\therefore d\underline{A} = \underline{\hat{n}}\, dA = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} dz\, dx$

$$\therefore \text{Flux}_{\text{Total}} = -4 - \frac{32}{3} = -4 - 10\tfrac{2}{3} = -14\tfrac{2}{3}$$

(ii) $\underline{\nabla} \cdot \underline{F} = 0 + 2(y+2x-4) - 2z = 2(y-z+2x-4).$

(iii) $$\int_V \underline{\nabla} \cdot \underline{F}\, dV = \int_{z=0}^{1}\int_{x=0}^{2}\int_{y=0}^{4-2x} 2(y-z+2x-4)\, dy\, dx\, dz$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{4-2x} y + 2x - 4 - \underbrace{\tfrac{1}{2}}_{=\int_0^1 -z\, dz}\, dy\, dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{4-2x} 2y + 4x - 9\, dy\, dx$$

$$= \int_{x=0}^{2} \left[y^2 + (4x-9)y\right]_{y=0}^{4-2x} dx$$

$$= \int_{x=0}^{2} (4-2x)^2 + (4x-9)(4-2x)\, dx$$

$$= \int_{x=0}^{2} 16 - \cancel{16x} + 4x^2 + \cancel{16x} - 8x^2 - 36 + 18x\, dx$$

$$= \int_{x=0}^{2} -20 + 18x - 4x^2\, dx$$

$$= \left[-20x + 9x^2 - \frac{4}{3}x^3\right]_0^2$$

$$= -40 + 36 - \frac{2^5}{3}$$

$$= -4 - \frac{32}{3}$$

$$= -4 - 10\tfrac{2}{3}$$

$$= -14\tfrac{2}{3}.$$

(iv) The answers to (i) & (iii) are the same, as expected by the divergence theorem: $\int_V \underline{F} \cdot d\underline{V} = \int_{\partial V} \underline{F} \cdot d\underline{S}$.

17Z

We can treat the following coupled system of differential equations as an eigenvalue problem:

$$2\frac{dy_1}{dt} = 2f_1 - 3y_1 + y_2,$$
$$2\frac{dy_2}{dt} = 2f_2 + y_1 - 3y_2,$$
$$\frac{dy_3}{dt} = f_3 - 4y_3,$$

where f_1, f_2 and f_3 is a set of time-dependent sources, and y_1, y_2 and y_3 is a set of time-dependent responses.

(a) If these equations are written using matrix notation,

$$\frac{d\mathbf{y}}{dt} + \mathbf{K}\mathbf{y} = \mathbf{f},$$

what are the elements of $\mathbf{K}$? Find the eigenvalues and eigenvectors of $\mathbf{K}$. [6]

(b) In the case when the system is not excited, $\mathbf{f} = \mathbf{0}$, find all of the solutions having the form

$$\mathbf{y}(t) = \mathbf{y}(0)e^{-\gamma t},$$

where $\gamma > 0$ is a decay constant. [4]

(c) If $\mathbf{f}$ is held constant at $\mathbf{f}_0$, the response vector $\mathbf{y}$ has the steady state value $\mathbf{y}_0$ (that is, with $\frac{d\mathbf{y}}{dt} = 0$). Write down $\mathbf{y}_0$ in terms of $\mathbf{f}_0$, and find $\mathbf{y}_0$ in the case where $\mathbf{f}_0 = (1, 1, 1)^T$. [6]

(d) Assume that $\mathbf{y}$ starts in the steady state solution $\mathbf{y}_0$ given in (c) with $\mathbf{f}_0 = (1, 1, 1)^T$. Now suppose the source function abruptly falls to zero, $\mathbf{f}_0 = (0, 0, 0)^T$, so that the response vector $\mathbf{y}$ moves away from $\mathbf{y}_0$. Writing $\mathbf{y}$ as a linear combination of the allowed solutions found in (b), derive an expression for the subsequent time evolution of the system. [4]

Solution(s):

From user: lester

(a) $\left\{\begin{array}{l} 2\frac{dy_1}{dt} = 2f_1 - 3y_1 + y_2 \\ 2\frac{dy_2}{dt} = 2f_2 + y_1 - 3y_2 \\ \frac{dy_3}{dt} = f_3 - 4y_3 \end{array}\right\} \Leftrightarrow \frac{d}{dt}\underset{\underset{\underline{y}}{\uparrow}}{\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}} + \underbrace{\begin{pmatrix} 3/2 & -\frac{1}{2} & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 4 \end{pmatrix}}_{K} = \underset{\underset{\underline{f}}{\uparrow}}{\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}}.$ By inspection, K has eigenvectors $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \& \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ with eigenvalues $1, 2$ & 4 respectively.

(b) If $\frac{d\underline{y}}{dt} + K\underline{y} = \underline{0}$ & $\underline{y} = \underline{y}_0 e^{-\gamma t}$ then $-\gamma \underline{y}_0 e^{-\gamma t} + K\underline{y}_0 e^{-\gamma t} = 0 \Rightarrow K\underline{y}_0 = \gamma \underline{y}_0 \Rightarrow \gamma$ and $\underline{y}_0$ are e-vals and e-vecs of $K \Rightarrow$ possible solutions are of form $\underline{y} = A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + B\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-2t} + C\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-4t}$ for constants $A, B \& C$. (*)

(c) If $K\underline{y}_0 = \underline{f}_0$, $\underline{y}_0 = K^{-1}\underline{f}_0$. $\left[K^{-1} = \begin{pmatrix} \frac{1}{\frac{9}{4}-\frac{1}{4}}\begin{pmatrix} 3/2 & 1/2 \\ \frac{1}{2} & 3/2 \end{pmatrix} & 0 \\ & 0 \\ 0 \quad 0 & 1/4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right]$

So if $\underline{f}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then $\underline{y}_0 = \frac{1}{4}\begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1/4 \end{pmatrix}$.

(d) We are asked to solve $\frac{d\underline{y}}{dt} + K\underline{y} = \underline{0}$ subject to $\underline{y}(0) = \begin{pmatrix} 1 \\ 1 \\ 1/4 \end{pmatrix}$, so we must find the $A, B \& C$ in (*) that achieve this.

Solve $\left\{\begin{array}{l} A + B = 1 \\ A - B = 1 \\ C = 1/4 \end{array}\right\} \Rightarrow A = 1,\ B = 0,\ C = 1/4 \Rightarrow \underline{y} = 1\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} e^{-t} + 0\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-2t} + \frac{1}{4}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-4t} = \begin{pmatrix} e^{-t} \\ e^{-t} \\ \frac{1}{4}e^{-4t} \end{pmatrix}$.

18S

(a) Suppose $f(x)$ is a 2π-periodic function defined on $-\pi \leqslant x < \pi$. Write down its Fourier series and give expressions for the coefficients appearing in it. Using the orthogonality relations or otherwise, determine the value of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2 \, dx$$

in terms of the Fourier coefficients of f (Parseval's identity). [7]

(b) Show that the Fourier series of the 2π-periodic function $g(x) = x^3 - \pi^2 x$ for $-\pi \leqslant x < \pi$ is given by

$$g(x) = 12\sum_{n=1}^{\infty} \frac{(-1)^n}{n^p} \sin nx \,,$$

where the integer p should be determined. [7]

(c) Using Parseval's identity for g, show that

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945} \,.$$

[6]

Solution(s):

From user: lester

(a) $f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ has period 2π.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

Integrate to zero (orthogonality)

$$\frac{1}{\pi}\int_{-\pi}^{\pi} (f(x))^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\Big(a_0^2 + \sum_n a_n^2 \underbrace{\cos^2 nx} + \sum_n b_n^2 \underbrace{\sin^2 nx} + a_0 \sum_n a_n \overbrace{\cos nx} + a_0 \sum_n b_n \overbrace{\sin nx} + \sum_{m,n} a_m b_n \overbrace{\cos mx \sin nx}\Big)\,dx$$

Integrate to π

$$= \frac{1}{\pi}\Big(2\pi a_0^2 + \pi\sum_n a_n^2 + \pi \sum_n b_n^2\Big) = 2a_0^2 + \sum_n (a_n^2 + b_n^2).$$

(b) $g(x) = x^3 - \pi^2 x$. $g(x)$ is odd, so $a_0 = a_n = 0 \;\; \forall n$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x^3 - \pi^2 x)\sin nx\,dx \;:\; I_k = \int_{-\pi}^{\pi} x^k \sin nx\,dx = \left[x^k \frac{\cos nx}{n}\right]_{\pi}^{-\pi} + \frac{k}{n}\int_{-\pi}^{\pi} x^{k-1}\cos nx\,dx$$

$$= -\frac{2\pi^k}{n}(-1)^n + \frac{k}{n}\left\{\left[\overset{0}{\cancel{x^{k-1}\frac{\sin nx}{n}}}\right]_{-\pi}^{\pi} - \frac{k-1}{n}\int_{-\pi}^{\pi} x^{k-2}\sin nx\,dx\right\}$$

$$= -\frac{2\pi^k}{n}(-1)^n - \frac{k(k-1)}{n^2}\int_{-\pi}^{\pi} x^{k-2}\sin nx\,dx$$

$$= -\frac{2\pi^k}{n}(-1)^n - \frac{k(k-1)}{n^2} I_{k-2}$$

$$= \frac{1}{\pi}(I_3 - \pi^2 I_1) = \frac{1}{\pi}\left(-\frac{2\pi^3}{n}(-1)^n - \left(\frac{6}{n^2} + \pi^2\right) I_1\right)$$

$$= -\frac{2\pi^2}{n}\cancel{(-1)^n} - \left(\frac{6}{\pi n^2} + \cancel{\pi}\right)\left(-\frac{2\pi}{n}(-1)^n - 0\right) = \frac{12(-1)^n}{n^3}$$

$$\therefore \; g(x) = 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin(nx). \qquad (\text{"}p\text{"} = 3)$$

By Parseval's Theorem, then,

$$\sum_{n=1}^{\infty}\left(\frac{12(-1)^n}{n^3}\right)^2 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^3 - \pi^2 x)^2\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^6 - 2\pi^2 x^4 + \pi^4 x^2\,dx = \frac{1}{\pi}\left[\frac{1}{7}x^7 - \frac{2}{5}\pi^2 x^5 + \frac{1}{3}\pi^4 x^3\right]_{-\pi}^{\pi}$$

$$= 2\pi^6\left(\frac{1}{7} - \frac{2}{5} + \frac{1}{3}\right) = 2\pi^6\,\frac{15 - 42 + 35}{105} = \frac{16\pi^6}{105}$$

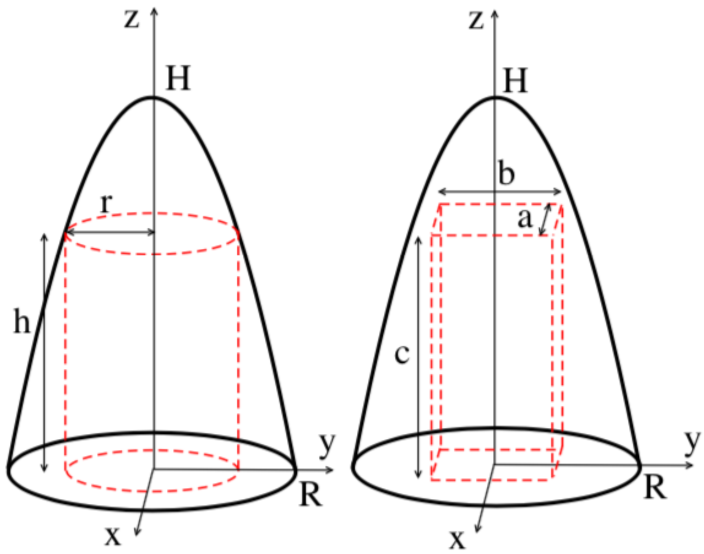
$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{16\pi^6}{105 \times 144} = \frac{4^2\pi^6}{105\cdot 4^2 \cdot 3^2} = \frac{\pi^6}{945}. \quad \text{QED.}$$

19*

The interior region of a paraboloid of height H and radius R of the base is defined by the following inequalities

$$0 < z < H\left[1 - (x^2 + y^2)/R^2\right] .$$

Either a cylinder of height h and radius r or a rectangular parallelepiped with sides a, b and c can be inscribed into the paraboloid as shown by dashed lines in the left and right panels of the diagram, respectively.



By using the method of Lagrange multipliers,

(a) show that the maximum possible volume of a cylinder, V_c, inscribed into the paraboloid as shown in the diagram above is

$$V_c = \frac{\pi R^2 H}{4} , \qquad [7]$$

(b) find in terms of H and R the maximum possible volume of the rectangular parallelepiped, V_p, inscribed into the paraboloid, [11]

(c) and thus determine which shape can produce a larger volume. [2]

[*Hint: You need not prove that suitable extrema you find are actually maxima.*]

Solution(s):

From user: lester

(a) $\mathcal{L}(r,h,\lambda) = \pi r^2 h + \lambda\left(H\left(1-\frac{r^2}{R^2}\right)-h\right)$

① $\frac{\partial \mathcal{L}}{\partial r} = 0 = 2\pi r h - 2\lambda H \frac{r}{R^2} \Rightarrow \underset{\text{silly}}{r=0}$, or $\pi h = \frac{\lambda H}{R^2}$

② $\frac{\partial \mathcal{L}}{\partial h} = 0 = \pi r^2 - \lambda \Rightarrow \lambda = \pi r^2$

③ $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow h = H\left(1-\frac{r^2}{R^2}\right)$

① & ② $\Rightarrow \cancel{\pi} h = \frac{\cancel{\pi} r^2 H}{R^2} \Rightarrow \frac{r^2}{R^2} = \frac{h}{H}$

(① & ②) + ③ $\Rightarrow \frac{h}{H} = 1 - \frac{h}{H} \Rightarrow \frac{h}{H} = \frac{1}{2} \Rightarrow h = \frac{1}{2}H$ & $r = \frac{1}{\sqrt{2}}R$.

$\Rightarrow V_c = \pi\left(\frac{1}{\sqrt{2}}R\right)^2\left(\frac{1}{2}H\right) = \frac{1}{4}\pi R^2 H$ Q.E.D.

(b) $\mathcal{L}(\bar{a},\bar{b},c,\lambda) = \bar{a}\bar{b}c + \frac{1}{2}\lambda\left(H\left(1-\frac{\bar{a}^2+\bar{b}^2}{R^2}\right)-c\right)$ (wlog $a,b,c>0$)

$\bar{a} = \frac{a}{2}$, $\bar{b} = \frac{b}{2}$

① $\frac{\partial \mathcal{L}}{\partial \bar{a}} = 0 \Rightarrow \bar{b}c = \frac{\lambda H \bar{a}}{R^2}$

② $\frac{\partial \mathcal{L}}{\partial \bar{b}} = 0 \Rightarrow \bar{a}c = \frac{\lambda H \bar{b}}{R^2}$

$+③ \Rightarrow \begin{bmatrix} R^2 \cancel{\bar{b}} c = 2\bar{a}^2 \cancel{\bar{b}} H \\ R^2 \cancel{\bar{a}} c = 2\cancel{\bar{a}}\bar{b}^2 H \end{bmatrix} \Rightarrow \begin{bmatrix} \bar{a} = \bar{b} \\ \& \\ 2\bar{a}^2 = \frac{R^2 c}{H} \end{bmatrix}$ ⑤ ⑥

③ $\frac{\partial \mathcal{L}}{\partial c} = 0 \Rightarrow \bar{a}\bar{b} = \frac{\lambda}{2}$

④ $\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow 1 - \frac{\bar{a}^2+\bar{b}^2}{R^2} = \frac{c}{H}$

④, ⑤ & ⑥ $\Rightarrow 1 - \frac{c}{H} = \frac{c}{H} \Rightarrow c = \frac{1}{2}H$

$\Rightarrow 2\bar{a}^2 = \frac{R^2}{2} \Rightarrow \bar{a} = \frac{R}{2} \Rightarrow a = b = R$

$\Rightarrow V_p = abc = \frac{1}{2}HR^2$ $\therefore$ $\frac{V_c}{V_p} = \frac{\pi}{4}\cdot 2 = \frac{\pi}{2} > 1 \Rightarrow \underline{\underline{V_c > V_p}}$

20*

(a) (i) Solve the equation

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{1+x},$$

subject to the boundary condition $y(0) = 1$. [4]

(ii) Solve the equation

$$\frac{dy}{dx} + \tfrac{1}{3}y = e^x y^4,$$

subject to the boundary condition $y(0) = 1$. [5]

(b) The following partial differential equation on the given interval,

$$\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}, \qquad 0 \leqslant x \leqslant L, \quad t \geqslant 0, \qquad (\ddagger)$$

has the boundary conditions $u(0,t) = u(L,t) = 0$. By using the separable function $u(x,t) = X(x)T(t)$, show that the equation (‡) may be written as

$$\frac{1}{T}\frac{dT}{dt} + 1 = \frac{1}{X}\frac{d^2X}{dx^2} = -k^2,$$

with k a constant.

Determine the functions $X(x), T(t)$ satisfying the boundary conditions.

Hence, write down the general solution of the partial differential equation (‡). [11]

Solution(s):

From user: niy20

20Y* (a) (i) $\frac{dy}{dx} = -\frac{\sqrt{y}}{1+x}$ → separable.

$\Rightarrow \frac{dy}{\sqrt{y}} = -\frac{dx}{1+x}$

$\Rightarrow \int \frac{dy}{\sqrt{y}} = \int -\frac{dx}{1+x}$

$\Rightarrow 2\sqrt{y} = -\ln|1+x| + C$ → arbitrary constant

$y(x=0) = 1$

$\Rightarrow 2\sqrt{1} = -\ln(1) + C \Rightarrow C = 2$

$\Rightarrow 2\sqrt{y} = -\ln|1+x| + 2 \Rightarrow \sqrt{y} = 1 - \ln(\sqrt{1+x})$

$\Rightarrow \boxed{y = (1 - \ln(\sqrt{1+x}))^2}$

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(ii) $\frac{dy}{dx} + \frac{1}{3}y = e^x y^4$ → Bernoulli ODE

Let $u = y^{1-4} = y^{-3} = \frac{1}{y^3}$

$\Rightarrow y = \sqrt[3]{\frac{1}{u}} = u^{-1/3}$

$\Rightarrow \frac{dy}{dx} = \left(-\frac{1}{3}\right) \cdot u^{-4/3} \cdot \frac{du}{dx}$

⇒ ODE becomes:

$-\frac{1}{3} u^{-4/3} \frac{du}{dx} + \frac{1}{3} \cdot u^{-1/3} = e^x u^{-4/3} \quad / \cdot -3 \cdot u^{4/3}$

$\Rightarrow \frac{du}{dx} - u = e^x$ → linear inhomogeneous

$\Rightarrow$ Let $\mu(x) = \exp\left(\int^x -1\, dx'\right) = \exp(-x)$

$\Rightarrow \frac{d(\mu(x) u)}{dx} = e^x \cdot e^{-x} = 1$

$\Rightarrow \mu(x) \cdot u = x + C$ → arbitrary constant

$\Rightarrow u = \frac{x + C}{\mu(x)} = (x+C)\exp(x)$

$y = u^{-1/3}$

$\Rightarrow \boxed{y = \left((x+C) \cdot \exp(x)\right)^{-1/3}}$

Boundary conditions : $y(0) = 1$

$\Rightarrow 1 = (C)^{-1/3} \Rightarrow C = 1 \quad (C \in \mathbb{R})$

$\Rightarrow \boxed{y = (x+1)^{-1/3} \exp\left(-\frac{x}{3}\right)}$

From user: niy20

(B) $\frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2}$, $x \in [0; L]$, $t \geq 0$

Boundary conditions:

$u(0,t) = 0 = u(L,t)$

Try separable solution: $u(x,t) = X(x) \cdot T(t)$

$\Rightarrow X \cdot T' + X \cdot T = X'' \cdot T \quad \Big| \cdot \frac{1}{X \cdot T}$ ($X, T \neq 0$ since that just yields trivial sol.)

$\Rightarrow \frac{T'}{T} + 1 = \frac{X''}{X} \quad (1)$

LHS and RHS of (1) depend on different, independent variables => they must be equal to a const.:

$\frac{T'}{T} + 1 = \frac{X''}{X} = \text{const}$

Due to boundary conditions, we want an oscillatory solution for $X(x)$ => choose const. < 0 ~~ie~~ e.g. const $= -k^2$

$$\Rightarrow \boxed{\frac{1}{T} \cdot \frac{dT}{dt} + 1 = \frac{1}{X} \cdot \frac{d^2 X}{dx^2} = -k^2}$$

$$\Rightarrow \left| \begin{array}{ll} X'' + k^2 X = 0 & \Rightarrow X_k(x) = A_k \cdot \cos(kx) + B_k \cdot \sin(kx) \\ T' + (1+k^2) T = 0 & \Rightarrow T_k(t) = C_k \cdot \exp(-(1+k^2)t) \end{array} \right.$$

$$\Rightarrow u_k(x,t) = \left(A_k \cdot \cos(kx) + B_k \sin(kx)\right) \cdot \exp(-(1+k^2)t)$$

Most general solution now is sum of all u_k.

Apply Boundary conditions to each of $u_k(x,t)$ for simplicity:

$$\begin{cases} u_k(0,t)=0 \Rightarrow A_k=0 \\ u_k(L,t)=0 \Rightarrow \sin(kL)=0 \Rightarrow kL = n\cdot\pi, \underbrace{n\in\mathbb{Z}^+}_{\substack{n=0 \text{ yields} \\ \text{trivial sol.} \\ u=0}} \\ \qquad \Rightarrow k=\frac{n\pi}{L} \end{cases}$$

$\Rightarrow$ Most general solution is:

$$\boxed{u(x,t)=\sum_{n=1}^{+\infty} B_n\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot\exp\left(-\frac{(L^2+n^2\pi^2)}{L^2}\cdot t\right)}$$