2016 Mathematics (1)

7A

Consider the mapping $z = f(\zeta)$ such that $G(z) = G(f(\zeta)) = \psi(\zeta)$, where f, G, ψ are complex functions and z, ζ are complex variables.

(a) What condition(s) must be satisfied for $\psi(\zeta)$ to be analytic? [3]

(b) Suppose that $\psi(\zeta) = \ln(\zeta + 2)$ and $f(\zeta)$ is defined by

$$\frac{df}{d\zeta} = \frac{i}{\sqrt{(\zeta+1)(\zeta-1)}}, \qquad (\star)$$

where $\zeta = 0$ maps to $z = 0$.

(i) By integrating $(\star)$, show that the upper half of the ζ plane maps onto the region R defined by $|\text{Re}(z)| \leqslant \frac{1}{2}\pi$, $\text{Im}(z) \geqslant 0$. Determine the location of any points in the region R where $G(z)$ is not analytic. How do these relate to points in the ζ plane? [*Hint:* $\sin(x + iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$.] [7]

(ii) The vector field $\mathbf{u} = (u, v)$ in the ζ plane is given by $u - iv = d\psi/d\zeta$. How does the magnitude of $\mathbf{u}$ vary across the upper half of the ζ plane? In what direction is $\mathbf{u}$ oriented? [3]

(iii) The vector field $\mathbf{U} = (U, V)$ is defined in the region R of the z plane by $U - iV = dG/dz$. Determine this field and use a sketch to illustrate the orientation of the vector field in this region. [7]

No soution has yet been submitted for this question.