2015 Mathematics (2)

Section A

1

The point A with position vector $(1,1,1)$ lies in a plane. The vectors $\boldsymbol{u} = (1,1,2)$ and $\boldsymbol{v} = (0,2,-1)$ are parallel to the same plane. Find $\hat{\boldsymbol{n}}$, the unit normal to the plane. Find the perpendicular distance, p, from the plane to the origin. [2]

2

Find all the roots of the equation $z^3 = -8$. [2]

3

Write down the first non-zero term of the Taylor series for the function $f(x) = \ln(x^2 + 1)$ about the origin, $x = 0$. Hence, or otherwise, state the type of stationary point at the origin. [2]

[You may quote the Taylor series expansions for standard functions.]

4

Find the eigenvalues and normalised eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

[2]

5

Let $\boldsymbol{F}$ be the gradient of $\Phi(x,y,z) = x\cos(y^5)\sinh z$. Find an expression for $\boldsymbol{F}$. What is the curl of $\boldsymbol{F}$? [2]

6

Let $r = x\,i + y\,j + z\,k$ be the position vector. Evaluate

(a) $\nabla(r \cdot r)$, [1]

(b) $\nabla \cdot (a\,r - b)$, where a is a constant real number and b is a constant vector. [1]

7

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x$$

for $x \geqslant 1$, given that $y = 1$ when $x = 1$. [2]

8

Given $F = y^2 x\,i + x^2 y\,j + \frac{1}{3}z^3\,k$, evaluate $\iiint \nabla \cdot F \, \mathrm{d}V$ inside a sphere of radius R, centred at the origin. [2]

[*Hint: You may find it helpful to work in spherical polar coordinates.*]

9

Find the Fourier sine series for $f(x) = \sin x(1 + 4\cos x)$ defined on $-\pi < x < \pi$. [2]

10

Ten fair coins are tossed simultaneously. Find expressions (which need not be evaluated) for

(a) the probability that ten heads are obtained,

(b) the probability that at least two coins give tails.

[2]

Section B

11R

(a) Two lines are defined by

$$\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \qquad \text{and} \qquad \boldsymbol{r} = 6\boldsymbol{j} + 4\boldsymbol{k} + \mu(-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}),$$

where $\boldsymbol{r}$ is the position vector, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the Cartesian unit vectors, and λ and μ are real parameters. Find the position vector, $\boldsymbol{p}$, of the point of intersection of the two lines, and the values of λ and μ at the point of intersection. [4]

(b) Solve the vector equation

$$\boldsymbol{r} + (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b} = \boldsymbol{c}$$

for the vector $\boldsymbol{r}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are constant vectors, in each of the following cases:

(i)
$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}, \qquad \boldsymbol{c} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k},$$
[4]

(ii)
$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \qquad \boldsymbol{c} = -2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},$$
[7]

(iii)
$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}, \qquad \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}, \qquad \boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k},$$
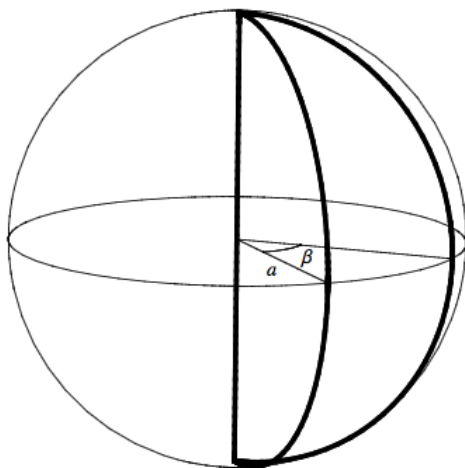[2]

and give a geometrical interpretation for each case. [3]

12S

The position vector of the centre of mass of a homogeneous solid body occupying a volume V is

$$\bar{\boldsymbol{x}} = \frac{1}{V} \int_V \boldsymbol{x}\, dV.$$

Let V be a wedge of angle β $(0 < \beta < 2\pi)$ taken from a solid sphere of radius a:

(a) Show that the centre of mass of V is located at a distance $a f(\beta)$ from the centre of the sphere, where

$$f(\beta) = \frac{3\pi}{8\beta} \sin(\tfrac{1}{2}\beta).$$

[10]

Sketch the graph of $f(\beta)$ for $0 < \beta < 2\pi$. [3]

(b) Calculate the vector area of the curved part of the surface of V. [7]

13T

The force fields $\boldsymbol{F}$ and $\boldsymbol{G}$ are given by

$$\boldsymbol{F} = \begin{pmatrix} xy \cosh z \\ x^2 \cosh z \\ x^2 y \sinh z \end{pmatrix}, \qquad \boldsymbol{G} = \begin{pmatrix} 2xy \cosh z \\ x^2 \cosh z \\ x^2 y \sinh z \end{pmatrix}.$$

(a) For each of the vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$, determine whether the vector field is conservative, and, if so, find a function Φ such that it is equal to $-\boldsymbol{\nabla}\Phi$. [6]

(b) Evaluate $\int \boldsymbol{F} \cdot d\boldsymbol{x}$ and $\int \boldsymbol{G} \cdot d\boldsymbol{x}$ along the path consisting of straight lines $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$. [6]

(c) Evaluate $\int \boldsymbol{F} \cdot d\boldsymbol{x}$ and $\int \boldsymbol{G} \cdot d\boldsymbol{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. [8]

14W

A device consists of two blocks. The time of failure of the first block, t_1, is uniformly distributed in the interval $0 < t_1 < T_1$. For $t \geqslant T_1$, the first block has certainly failed. The time of failure of the second block, t_2, is linearly distributed in the interval $0 < t_2 < T_2$ according to the probability density function $f_2(t_2) = At_2$. For $t \geqslant T_2$, the second block has certainly failed.

(a) What is the probability density function, $f_1(t_1)$, of the time of failure of the first block? [2]

(b) Find A and sketch $f_2(t_2)$ for $0 < t_2 < \infty$. [2]

(c) Find and sketch $P_1(t)$, the probability that the first block fails at any time less than t, where $0 < t < \infty$. [3]

(d) Find and sketch $P_2(t)$, the probability that the second block fails at any time less than t, where $0 < t < \infty$. [3]

Assume from now on that $T_1 = T_2 = T$.

(e) Find and sketch $P(t)$, the probability that both blocks have failed by time t, where $0 < t < \infty$. [4]

(f) Find and sketch $R(t)$, the probability that at least one of the blocks has failed by time t, where $0 < t < \infty$. Mark on your graph the inflexion point(s) (if any) and calculate their coordinates. [6]

15Y

(a) The function $f(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + 8\frac{\mathrm{d}f}{\mathrm{d}t} + 12f = 12\,\mathrm{e}^{-4t}.$$

For the following sets of boundary conditions determine whether the equation has solutions consistent with all three conditions and, if so, find those solutions.

(i) $f(0) = 0$, $\dfrac{\mathrm{d}f}{\mathrm{d}t}(0) = 0$, $f(\ln\sqrt{2}) = 0$,

(ii) $f(0) = 0$, $\dfrac{\mathrm{d}f}{\mathrm{d}t}(0) = -2$, $f(\ln\sqrt{2}) = 0$. [10]

(b) A solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4\,\mathrm{e}^{-x}$$

takes the value 1 when $x = 0$ and the value e^{-1} when $x = 1$. What is its value when $x = 2$? [10]

16Y

(a) State the condition on the partial derivatives of P and Q for the differential form

$$P(x,y)\,\mathrm{d}x + Q(x,y)\,\mathrm{d}y$$

to be exact. If this condition is not satisfied, show that the differential form can be made exact by multiplying by an integrating factor of the form $\mu(x)$, provided that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

is a function of x only. What is the corresponding condition for the differential form to have an integrating factor of the form $\mu(y)$? [5]

Solve the following differential equations using an integrating factor:

(i)
$$2x\frac{\mathrm{d}y}{\mathrm{d}x} + 3x + y = 0,$$

[5]

(ii)
$$(\cos^2 x + y\sin 2x)\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0.$$

[5]

(b) Use the change of variables $y(x) = u(x)x$ to solve the differential equation

$$(y-x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2x + 3y = 0.$$

[5]

17Z

(a) In the context of matrices, describe what is meant by an eigenvector equation. [3]

Four springs, each having stiffness constant k, are used to hold three masses m in a line between two rigid supports. It can be shown that the dynamical behaviour of the system is described by the following three coupled differential equations:

$$m\ddot{x}_1 = -kx_1 + k(x_2 - x_1),$$
$$m\ddot{x}_2 = -k(x_2 - x_1) + k(x_3 - x_2),$$
$$m\ddot{x}_3 = -k(x_3 - x_2) - kx_3,$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the positions of the masses, along the line of the springs, relative to their static positions, and the double dot denotes the second derivative with respect to time t.

(b) Show that the substitutions

$$x_i = a_i \cos(\omega t), \qquad i = 1, 2, 3,$$

where a_i is independent of t, transform the differential equations into a set of simultaneous algebraic equations. [2]

(c) Write down a single matrix equation, in the form of an eigenvector equation, that describes the system. [3]

(d) Find the eigenvalues and normalised eigenvectors that satisfy this equation. [6]

(e) Show that the eigenvectors are orthogonal. [3]

(f) For the solution with the largest value of $|\omega|$, sketch x_1, x_2 and x_3 versus time. [3]

18Z

A certain electrical circuit produces a voltage waveform $V_1(t)$ that is periodic in time. The waveform can be written in terms of the Fourier series

$$V_1(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t). \qquad (\dagger)$$

(a) If the period of the waveform is T, write down an expression for ω. [1]

(b) Starting with ($\dagger$), demonstrate that the coefficients a_n and b_n are given by

$$a_n = \frac{2}{T} \int_0^T V_1(t) \cos(n\omega t)\, dt,$$

$$b_n = \frac{2}{T} \int_0^T V_1(t) \sin(n\omega t)\, dt.$$

[6]

(c) Suppose that the periodic voltage has the form $V_1(t) = V_0 \sin(\omega t)$ for $0 \leqslant t \leqslant T/2$ and $V_1(t) = 0$ for $T/2 < t < T$. Derive expressions for the Fourier coefficients, and write out the first 5 non-zero terms (in order of increasing frequency) of the series expansion explicitly. [8]

(d) Create a new function $V_2(t)$ by shifting $V_1(t)$ in time by $T/2$. By noting that $V_1(t) - V_2(t) = V_0 \sin(\omega t)$, and without evaluating the Fourier integrals explicitly, write out the first 5 non-zero terms of the series expansion of $V_2(t)$. [5]

19S*

(a) Explain (without proof) how the method of Lagrange multipliers is used to find the stationary points of the function $f(x, y)$ subject to the constraint $g(x, y) = c$, where c is a constant. How is the method generalized to handle a function of more than two variables subject to one or more constraints? [6]

(b) The function $f(x_1, x_2, x_3, \ldots, x_n)$ of n variables is defined by

$$f = -\sum_{i=1}^{n} x_i \ln x_i \,,$$

where the variables x_i are positive and subject to the constraint

$$\sum_{i=1}^{n} x_i = 1 \,.$$

Show that the stationary point of f subject to this constraint is located where $x_i = 1/n$ for each i, and calculate the stationary value of f. [7]

(c) If a further constraint

$$\sum_{i=1}^{n} x_i y_i = Y$$

is applied, where $y_1, y_2, y_3, \ldots, y_n$ and Y are given constants, show that the stationary point of the same function f is located instead where

$$x_i = a \, \exp(-b y_i) \,,$$

where a and b are constants. Write down two equations that determine the values of a and b. [7]

20T*

A fluid flows with velocity $u(x,t)$ along a channel bounded by walls at $x = 0$ and $x = 1$. The fluid velocity satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + G\sin(n\pi x)\,, \tag{\dagger}$$

where $\nu > 0$ and G are real constants and $n \geqslant 1$ is an integer. It also satisfies the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = 0$.

(a) For sufficiently large time t the fluid velocity tends to a steady-state solution $u_\mathrm{s}(x)$, independent of t, that satisfies equation ($\dagger$) and the same boundary conditions. Find $u_\mathrm{s}(x)$. [3]

(b) Now consider
$$\tilde{u}(x,t) = u(x,t) - u_\mathrm{s}(x)\,.$$

Find the partial differential equation, analogous to equation ($\dagger$), satisfied by $\tilde{u}(x,t)$, and show that this equation is independent of G. What are the boundary conditions that $\tilde{u}$ must satisfy, and what is the initial condition for $\tilde{u}$? [4]

(c) By means of separation of variables, find a solution for $\tilde{u}$ of the form

$$\tilde{u}(x,t) = f(t)g(x)\,,$$

where $\lim\limits_{t\to\infty} f(t) = 0$. [10]

(d) Hence determine the fluid velocity $u(x,t)$ and the total flow rate

$$Q(t) = \int_0^1 u(x,t)\,\mathrm{d}x\,.$$

[3]