# 2015 Mathematics (2)

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## Section A

### 1

The point $A$ with position vector $(1,1,1)$ lies in a plane. The vectors $\boldsymbol{u} = (1,1,2)$ and $\boldsymbol{v} = (0,2,-1)$ are parallel to the same plane. Find $\hat{\boldsymbol{n}}$, the unit normal to the plane. Find the perpendicular distance, $p$, from the plane to the origin. [2]

No soution has yet been submitted for this question.

### 2

Find all the roots of the equation $z^3 = -8$. [2]

No soution has yet been submitted for this question.

### 3

Write down the first non-zero term of the Taylor series for the function $f(x) = \ln(x^2+1)$ about the origin, $x = 0$. Hence, or otherwise, state the type of stationary point at the origin. [2]

[You may quote the Taylor series expansions for standard functions.]

No soution has yet been submitted for this question.

### 4

Find the eigenvalues and normalised eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

[2]

No soution has yet been submitted for this question.

### 5

Let $\boldsymbol{F}$ be the gradient of $\Phi(x,y,z) = x\cos(y^5)\sinh z$. Find an expression for $\boldsymbol{F}$. What is the curl of $\boldsymbol{F}$? [2]

No soution has yet been submitted for this question.

### 6

Let $\boldsymbol{r} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k}$ be the position vector. Evaluate

(a) $\boldsymbol{\nabla}(\boldsymbol{r}\cdot\boldsymbol{r})$, [1]

(b) $\boldsymbol{\nabla}\cdot(a\,\boldsymbol{r} - \boldsymbol{b})$, where $a$ is a constant real number and $\boldsymbol{b}$ is a constant vector. [1]

No soution has yet been submitted for this question.

## 7

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = x$$

for $x \geqslant 1$, given that $y = 1$ when $x = 1$. [2]

### *Solution(s):*

From user: lester

$$\frac{dy}{dx} - \frac{y}{x} = x. \quad (*)$$

$$I = e^{\int -\frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}.$$

$\therefore \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 1$ is interesting.

$$\Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = 1$$

$$\Rightarrow \frac{y}{x} = x + c.$$

BC ($y=1$ @ $x=1$) $\Rightarrow c = 0$.

$$\therefore y = x^2.$$

Check: $y = x^2 \Rightarrow \frac{dy}{dx} = 2x$.

$\therefore$ LHS $(*)$ $= 2x - x = x$ ✓

## 8

Given $\boldsymbol{F} = y^2 x\,\boldsymbol{i} + x^2 y\,\boldsymbol{j} + \frac{1}{3}z^3\,\boldsymbol{k}$, evaluate $\iiint \boldsymbol{\nabla}\cdot\boldsymbol{F}\,\mathrm{d}V$ inside a sphere of radius $R$, centred at the origin. [2]

[*Hint: You may find it helpful to work in spherical polar coordinates.*]

No soution has yet been submitted for this question.

## 9

Find the Fourier sine series for $f(x) = \sin x(1 + 4\cos x)$ defined on $-\pi < x < \pi$. [2]

***Solution(s):***

From user: lester

$$(1+4\cos x)\sin x = \sin x + 2\sin 2x$$

## 10

Ten fair coins are tossed simultaneously. Find expressions (which need not be evaluated) for

(a) the probability that ten heads are obtained,

(b) the probability that at least two coins give tails.

[2]

***Solution(s):***

From user: lester

(a) $P(10\text{ heads}) = \left(\frac{1}{2}\right)^{10}$

(b) $P(2\text{ or more tails}) = 1 - P(1\text{ or fewer tails})$

$$= 1 - \left(\left(\tfrac{1}{2}\right)^{10} + 10\left(\tfrac{1}{2}\right)^{9}\left(\tfrac{1}{2}\right)^{1}\right) = 1 - 11\left(\tfrac{1}{2}\right)^{10}$$

## Section B

## 11R

(a) Two lines are defined by

$$\boldsymbol{r} = 2\boldsymbol{i} - \boldsymbol{j} + \lambda(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \qquad \text{and} \qquad \boldsymbol{r} = 6\boldsymbol{j} + 4\boldsymbol{k} + \mu(-\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k})\,,$$

where $\boldsymbol{r}$ is the position vector, $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the Cartesian unit vectors, and $\lambda$ and $\mu$ are real parameters. Find the position vector, $\boldsymbol{p}$, of the point of intersection of the two lines, and the values of $\lambda$ and $\mu$ at the point of intersection. [4]

(b) Solve the vector equation

$$\boldsymbol{r} + (\boldsymbol{a} \cdot \boldsymbol{r})\boldsymbol{b} = \boldsymbol{c}$$

for the vector $\boldsymbol{r}$, where $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are constant vectors, in each of the following cases:

(i)

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,, \qquad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\,, \qquad \boldsymbol{c} = 3\boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}\,,$$

[4]

(ii)

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,, \qquad \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}\,, \qquad \boldsymbol{c} = -2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,,$$

[7]

(iii)

$$\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,, \qquad \boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}\,, \qquad \boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\,,$$

[2]

and give a geometrical interpretation for each case. [3]

## *Solution(s):*

From user: lester

(a) At intersection:

$$\underline{p} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\6\\4\end{pmatrix} + \mu\begin{pmatrix}-1\\2\\1\end{pmatrix}$$

Top Line: $2+\lambda = -\mu$

Bottom: $\lambda = 4+\mu$

$2+2\lambda = 4 \Rightarrow \lambda = 1$

$\Rightarrow \mu = -3.$

$$\therefore \underline{p} = \begin{pmatrix}2\\-1\\0\end{pmatrix} + \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\0\\1\end{pmatrix}$$

(b) $\underline{r} + (\underline{a}\cdot\underline{r})\underline{b} = \underline{c}$ $(+)$

$\Rightarrow \underline{r} = \underline{c} - (\underline{a}\cdot\underline{r})\underline{b}$

$\Rightarrow \underline{a}\cdot\underline{r} = \underline{a}\cdot\underline{c} - (\underline{a}\cdot\underline{r})(\underline{b}\cdot\underline{a})$

$\Rightarrow \underline{a}\cdot\underline{r}(1+\underline{a}\cdot\underline{b}) = \underline{a}\cdot\underline{c}$ $(*)$

$\Rightarrow \underline{a}\cdot\underline{r} = \dfrac{\underline{a}\cdot\underline{c}}{1+\underline{a}\cdot\underline{b}}$ if $\underline{a}\cdot\underline{b} \neq -1$

$\Rightarrow \underline{r} = \underline{c} - \dfrac{\underline{a}\cdot\underline{c}}{1+\underline{a}\cdot\underline{b}}\underline{b}$ if $\underline{a}\cdot\underline{b} \neq -1$.

But if $\underline{a}\cdot\underline{b} = -1$, then $(*)$ reads $0 = \underline{a}\cdot\underline{c}$. This may or may not be true, depending on the $\underline{a}$ and $\underline{c}$ supplied. If the supplied $\underline{a}$ and $\underline{c}$ do not have $\underline{a}\cdot\underline{c} = 0$, there are no solutions since we have a contradiction that stems from assuming that there is a solution. If $\underline{a}\cdot\underline{c}$ actually $\underline{\underline{\text{is}}}$ zero, then $(*)$ is uninformative.

(i) $\underline{a} = (1,1,1)$, $\underline{b} = (2,1,2)$ $\Big\}$ $\underline{a}\cdot\underline{b} = 5 \neq 1$ $\therefore$ $\underline{a}\cdot\underline{r} = \dfrac{6}{1+5} = 1$

$\underline{c} = (3,1,2)$ $\Rightarrow \underline{r} = \underline{c} - \underline{b} = (1,0,0)$.

(ii) $\underline{a} = (1,1,1)$, $\underline{b} = (1,-1,-1)$ $\Big\}$ $\underline{a}\cdot\underline{b} = -1$

$\underline{c} = (-2,1,1)$ $\underline{a}\cdot\underline{c} = 0 \Rightarrow$ $(*)$ is uninformative. Can $\underline{a}\cdot\underline{r}$ be free?

Try $\underline{r} = \underline{c} - \lambda\underline{b}$ ($\lambda$ free)

Then $\underline{r}\cdot\underline{a} = \underline{a}\cdot\underline{c} - \lambda\underline{b}\cdot\underline{a} = 0+\lambda = \lambda$

$\therefore$ $\underline{r} + (\underline{a}\cdot\underline{r})\underline{b} = \underline{c} - \lambda\underline{b} + \lambda\underline{b} = \underline{c}$ as required by $(+)$. So, indeed, $\underline{r} = \underline{c} - \lambda\underline{b}$ solves $(+)$ for any $\lambda$.

(iii) $\underline{a} = (1,1,1)$, $\underline{b} = (1,-1,-1)$ $\Big\}$ $\underline{a}\cdot\underline{b} = -1$

$\underline{c} = (-1,1,1)$ $\underline{a}\cdot\underline{c} = 1$ $\Big\} \Rightarrow$ $\times\!\!\times$ (contradiction!) $\Rightarrow$ No soln for $\underline{r}$.

(i) is a single point solution

(ii) is a line of solutions

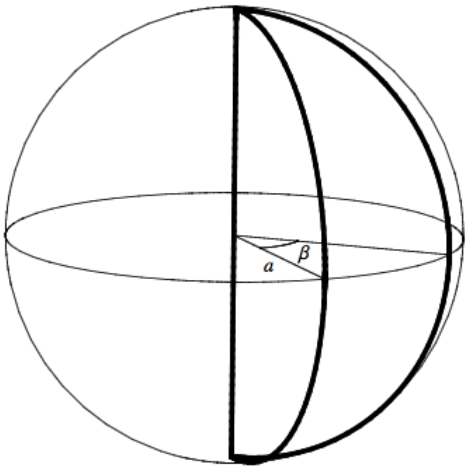
(iii) is an "empty set" of solutions.

**12S**

The position vector of the centre of mass of a homogeneous solid body occupying a volume $V$ is

$$\overline{\boldsymbol{x}} = \frac{1}{V}\int_V \boldsymbol{x}\,\mathrm{d}V\,.$$

Let $V$ be a wedge of angle $\beta$ ($0 < \beta < 2\pi$) taken from a solid sphere of radius $a$:




(a) Show that the centre of mass of $V$ is located at a distance $af(\beta)$ from the centre of the sphere, where

$$f(\beta) = \frac{3\pi}{8\beta}\sin(\tfrac{1}{2}\beta)\,.$$

[10]

Sketch the graph of $f(\beta)$ for $0 < \beta < 2\pi$. [3]

(b) Calculate the vector area of the curved part of the surface of $V$. [7]

## ***Solution(s):***

From user: rnp28

η5 a) $\underline{\bar{x}} = \frac{1}{V}\int_V \underline{x}\,dV \qquad V = \frac{\beta}{2\pi}\frac{4}{3}\pi a^3 \Rightarrow \frac{1}{V} = \frac{3}{2\beta a^3}$

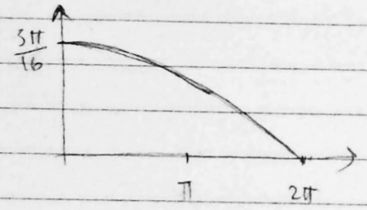
$\underline{x} = x\underline{i} + y\underline{j} + z\underline{k} = r\sin\theta\cos\phi\,\underline{i} + r\sin\theta\sin\phi\,\underline{j} + r\cos\theta\,\underline{k}$ in polar coordinates.

$$\int_V \underline{x}\,dV = \int_{\phi=-\frac{\beta}{2}}^{\beta/2}\int_{\theta=0}^{\pi}\int_{r=0}^{a}\Big(r\sin\theta\cos\phi\,\underline{i} + \underbrace{\cancel{r\sin\theta\sin\phi\,\underline{j}}}_{\substack{0 \text{ between} \\ -\frac{\beta}{2} \text{ and } \frac{\beta}{2}}} + \underbrace{\cancel{r\cos\theta\,\underline{k}}}_{\substack{0 \text{ between} \\ 0 \text{ and } \pi}}\Big) r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_{\phi=-\frac{\beta}{2}}^{\beta/2}\int_{\theta=0}^{\pi}\int_{r=0}^{a} r^3\sin^2\theta\cos\phi\,dr\,d\theta\,d\phi$$

$$= \left(2\sin\frac{\beta}{2}\right)\left(\frac{\pi}{2}\right)\left(\frac{1}{4}a^4\right) = \frac{\pi a^4\sin\frac{1}{2}\beta}{4}$$

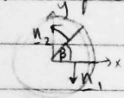
$$\Rightarrow \underline{\bar{x}} = \frac{3\pi a^{4}\sin\frac{1}{2}\beta}{8\beta a^3} = a\left(\frac{3\pi}{8\beta}\sin\left(\tfrac{1}{2}\beta\right)\right)$$




$f(\beta) = \frac{3\pi}{8}\frac{\sin c(\frac{1}{2}\beta)}{2}$

b) Vector area of curved surface = −(vector area of flat surfaces)

area of each flat surface $= \frac{\pi a^2}{2}$




wedge is symmetric about x-y plane so normals have 0 z coordinate

can define wedge position s.t. $\underline{n}_1 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ so $\underline{n}_2 = \begin{pmatrix}-\sin\beta\\\cos\beta\\0\end{pmatrix}$

so vector area of flat surfaces $= \pi a^2\left(\begin{pmatrix}0\\-1\\0\end{pmatrix} + \begin{pmatrix}-\sin\beta\\\cos\beta\\0\end{pmatrix}\right)$

$$= \pi a^2\begin{pmatrix}-\sin\beta\\-1+\cos\beta\\0\end{pmatrix}$$

vector area of curved surface $= \pi a^2\begin{pmatrix}\sin\beta\\1-\cos\beta\\0\end{pmatrix}$

$= \pi a^2\,|\sin\beta|$ in direction shown

From user: cgl20
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moo

## 13T

The force fields $\boldsymbol{F}$ and $\boldsymbol{G}$ are given by

$$\boldsymbol{F} = \begin{pmatrix} xy\cosh z \\ x^2\cosh z \\ x^2 y\sinh z \end{pmatrix}, \qquad \boldsymbol{G} = \begin{pmatrix} 2xy\cosh z \\ x^2\cosh z \\ x^2 y\sinh z \end{pmatrix}.$$

(a) For each of the vector fields $\boldsymbol{F}$ and $\boldsymbol{G}$, determine whether the vector field is conservative, and, if so, find a function $\Phi$ such that it is equal to $-\boldsymbol{\nabla}\Phi$. [6]

(b) Evaluate $\int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x}$ and $\int \boldsymbol{G}\cdot \mathrm{d}\boldsymbol{x}$ along the path consisting of straight lines $(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1)$. [6]

(c) Evaluate $\int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{x}$ and $\int \boldsymbol{G}\cdot \mathrm{d}\boldsymbol{x}$ along the straight line from $(0,0,0)$ to $(1,1,1)$. [8]

### *Solution(s):*

From user: cgl20

$$\underline{H} = \begin{pmatrix} \alpha xy\cosh z \\ x^2\cosh z \\ x^2 y\sinh z \end{pmatrix} \qquad \underline{F} = \underline{H}\big|_{\alpha=1}, \quad \underline{G} = \underline{H}\big|_{\alpha=2}$$

(a)
$$\underline{\nabla}\wedge\underline{H} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha xy\cosh z & x^2\cosh z & x^2 y\sinh z \end{vmatrix}$$

$$= \begin{pmatrix} x^2\sinh z - x^2\sinh z \\ \alpha xy\sinh z - 2xy\sinh z \\ 2x\cosh z - \alpha x\cosh z \end{pmatrix} = (\alpha-2)x\begin{pmatrix} 0 \\ y\sinh z \\ -\cosh z \end{pmatrix}$$

This factor is zero for $\underline{G}$ but not for $\underline{H}$.

$\therefore$ $\underline{\nabla}\wedge\underline{G} = \underline{0} \neq \underline{\nabla}\wedge\underline{F}$ so $\underline{F}$ is not conservative.

For $\underline{G}$: by inspection $\underline{G} = -\underline{\nabla}\big(\underbrace{-x^2 y\cosh z}_{\Phi}\big)$, and so $\underline{G}$ is conservative.

(b) Since $\underline{G}$ is conservative, expect to find that its line integrals depend only on the end points. Nonetheless, we may as evaluate explicit line integrals for both $\underline{F}$ & $\underline{G}$ since (i) we can do both at same time using $\underline{H}$, & (ii) it may be what the examiner wants.

Along $\Gamma_1 = (0,0,0) \xrightarrow{1} (1,0,0) \xrightarrow{2} (1,1,0) \xrightarrow{3} (1,1,1)$ have three lines:

Using $t\in[0,1]$ as the parameter for each of 1, 2 & 3 we find

$$\underline{H}\Big|_{1} = \begin{pmatrix} 0 \\ t^2 \\ 0 \end{pmatrix}, \quad \underline{H}\Big|_{2} = \begin{pmatrix} \alpha t \\ 1 \\ 0 \end{pmatrix}, \quad \underline{H}\Big|_{3} = \begin{pmatrix} \alpha\cosh t \\ \cosh t \\ \sinh t \end{pmatrix},$$

$$\therefore \int_{\Gamma_1} \underline{H}\cdot d\underline{x} = \int_{t=0}^{1} \begin{pmatrix} 0 \\ t^2 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} dt \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha t \\ 1 \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ dt \\ 0 \end{pmatrix} + \begin{pmatrix} \alpha\cosh t \\ \cosh t \\ \sinh t \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 0 \\ dt \end{pmatrix} dt$$

$$= \int_0^1 (1+\sinh t)\,dt = \Big[t+\cosh t\Big]_0^1$$

$$= 1 + \cosh 1 - (0+1) = \cosh 1.$$

This is independent of $\alpha$ so, $\int_{\Gamma_1}\underline{F}\cdot d\underline{x} = \int_{\Gamma_1}\underline{G}\cdot d\underline{x} = \cosh 1$.

(c) Along $\Gamma_2 = (0,0,0)\to(1,1,1)$ we have:

$$\int_{\Gamma_2}\underline{H}\cdot d\underline{t} = \int_{t=0}^{1}\begin{pmatrix} \alpha t^2\cosh t \\ t^2\cosh t \\ t^3\sinh t \end{pmatrix}\cdot\begin{pmatrix} dt \\ dt \\ dt \end{pmatrix} = \int_0^1 (\alpha+1)t^2\cosh t + t^3\sinh t \; dt$$

$$= \int_0^1 (\alpha+1)t^2\cosh t\,dt + \Big[t^3\cosh t\Big]_0^1 - \int_0^1 3t^2\cosh t\,dt$$

$$= \cosh 1 + (\alpha-2)\int_0^1 t^2\cosh t\,dt$$

Aside: $\int_0^1 t^2\cosh t\,dt = \Big[t^2\sinh t\Big]_0^1 - \int_0^1 2t\sinh t\,dt$

$= \sinh 1 - 2\Big[t\cosh t\Big]_0^1 + 2\int_0^1 \cosh t\,dt$

$= \sinh 1 - 2\cosh 1 + 2\Big[\sinh t\Big]_0^1$

$= \sinh 1 - 2\cosh 1 + (2\sinh 1) = \underline{\underline{3\sinh 1 - 2\cosh 1}}$

$\therefore \int_{\Gamma_2}\underline{H}\cdot d\underline{t} = \cosh 1 + (\alpha-2)(3\sinh 1 - 2\cosh 1)$.

$\therefore \int_{\Gamma_2}\underline{F}\cdot d\underline{t} = \cosh 1 - (3\sinh 1 - 2\cosh 1)$

$= 3(\cosh 1 - \sinh 1) = \frac{3}{e}$

& $\int_{\Gamma_2}\underline{G}\cdot d\underline{t} = \cosh 1$.

Let us make the check that, in the case of $\underline{G}$, both answers just obtained should equal $\Phi\big|_{0,0,0} - \Phi\big|_{1,1,1}$ which is $(-0^2 0\cosh 0) - (-1^2 1\cosh 1)$

$= \cosh 1$, as expected!

A device consists of two blocks. The time of failure of the first block, $t_1$, is uniformly distributed in the interval $0 < t_1 < T_1$. For $t \geqslant T_1$, the first block has certainly failed. The time of failure of the second block, $t_2$, is linearly distributed in the interval $0 < t_2 < T_2$ according to the probability density function $f_2(t_2) = At_2$. For $t \geqslant T_2$, the second block has certainly failed.

(a) What is the probability density function, $f_1(t_1)$, of the time of failure of the first block? [2]

(b) Find $A$ and sketch $f_2(t_2)$ for $0 < t_2 < \infty$. [2]

(c) Find and sketch $P_1(t)$, the probability that the first block fails at any time less than $t$, where $0 < t < \infty$. [3]

(d) Find and sketch $P_2(t)$, the probability that the second block fails at any time less than $t$, where $0 < t < \infty$. [3]

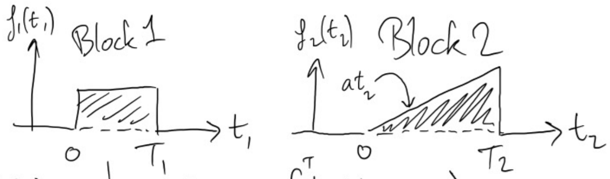
Assume from now on that $T_1 = T_2 = T$.

(e) Find and sketch $P(t)$, the probability that both blocks have failed by time $t$, where $0 < t < \infty$. [4]

(f) Find and sketch $R(t)$, the probability that at least one of the blocks has failed by time $t$, where $0 < t < \infty$. Mark on your graph the inflexion point(s) (if any) and calculate their coordinates. [6]

## *Solution(s):*

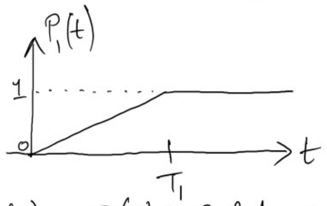
From user: lester



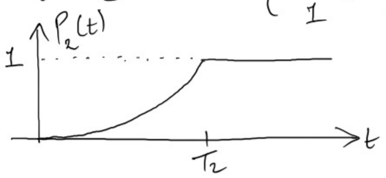

(a) $f_1(t_1) = \frac{1}{T_1}$ (so that $\int_0^T f_1(t_1)\,dt_1 = 1$)

(b) Want $1 = \int_0^{T_2} a t_2\,dt_2 = \left[\frac{1}{2} a t_2^2\right]_0^{T_2} = \frac{1}{2} a T_2^2 \Rightarrow a = \frac{2}{T_2^2}$.

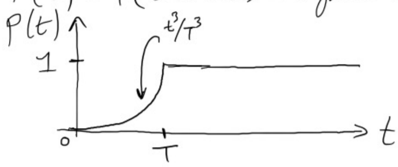
(c) $P_1(t) = P(\text{Block 1 fails at time} \le t) = \int_0^t \frac{1}{T_1}\,dt_1 = \frac{t}{T_1}$. This answer has assumed $0 \le t \le T_1$. More generally: $P_1(t) = \begin{cases} 0 & t \le 0 \\ t/T_1 & 0 \le t \le T_1 \\ 1 & t \ge T_1 \end{cases}$.




(d) $P_2(t) = P(\text{Block 2 fails at time} \le t) = \int_0^t \frac{2t_2}{T_2^2}\,dt_2 = \frac{t^2}{T_2^2}$ or, more generally: $P_2(t) = \begin{cases} 0 & t \le 0 \\ t^2/T_2^2 & 0 \le t \le T_2 \\ 1 & t \ge T_2 \end{cases}$.




(e) $P(t) = P(\text{both blocks have failed before time } t) = P_1(t)P_2(t) = \frac{t^3}{T^3}$ :




(f) $R(t) = P(\text{at least one block has failed}) = 1 - P(\text{neither has failed})$

$= 1 - (1 - P_1(t))(1 - P_2(t))$

$= 1 - \left(1 - \frac{t}{T}\right)\left(1 - \frac{t^2}{T^2}\right) = \frac{t}{T} + \frac{t^2}{T^2} - \frac{t^3}{T^3}$.

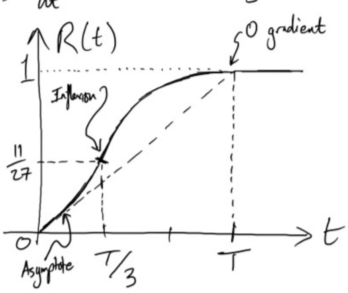
$\frac{dR}{dt} = \frac{1}{T} + \frac{2t}{T^2} - \frac{3t^2}{T^3} \Rightarrow \left.\frac{dR}{dt}\right|_{t=0} = \frac{1}{T}, \quad \left.\frac{dR}{dt}\right|_{t=T} = 0$

$\frac{d^2R}{dt^2} = \frac{2}{T^2} - \frac{6t}{T^3}$ At inflexions, $\frac{d^2R}{dt^2} = 0 \Rightarrow t = \frac{T}{3}$.

Putting these facts together;

$R\left(\frac{T}{3}\right) = \frac{1}{3} + \frac{1}{3^2} - \frac{1}{3^3}$

$= \frac{9+3-1}{27} = \frac{11}{27} > \frac{1}{3}$




15Y

(a) The function $f(t)$ satisfies the differential equation

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + 8\frac{\mathrm{d}f}{\mathrm{d}t} + 12f = 12\,\mathrm{e}^{-4t}\,.$$

For the following sets of boundary conditions determine whether the equation has solutions consistent with all three conditions and, if so, find those solutions.

(i) $f(0) = 0,\ \dfrac{\mathrm{d}f}{\mathrm{d}t}(0) = 0,\ f(\ln\sqrt{2}) = 0,$

(ii) $f(0) = 0,\ \dfrac{\mathrm{d}f}{\mathrm{d}t}(0) = -2,\ f(\ln\sqrt{2}) = 0.$ [10]

(b) A solution of the differential equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 4\,\mathrm{e}^{-x}$$

takes the value 1 when $x = 0$ and the value $\mathrm{e}^{-1}$ when $x = 1$. What is its value when $x = 2$? [10]

## *Solution(s):*

From user: ar857

2015 II 1&4

a) $f'' + 8f' + 12f = 12e^{-4t}$

$\lambda^2 + 8\lambda + 12 = 0 \Rightarrow \lambda_1 = -6 \quad \lambda_2 = -2$

$y_c = c_1 e^{-6t} + c_2 e^{-2t}$

$y_p = k e^{-4t}$

$Ly_p = ke^{-4t}(16-32+12) = ke^{-4t} \cdot -4 = 12e^{-4t} \Rightarrow k = -3$

$y = y_p + y_c = c_1 e^{-6t} + c_2 e^{-2t} - 3e^{-4t}$

$\left.\begin{aligned} y(0) &= c_1 + c_2 - 3 = 0 \\ y'(0) &= -6c_1 - 2c_2 + 12 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 3/2 \\ c_2 &= 3/2 \end{aligned}$

$y(\ln\sqrt{2}) = \frac{3}{2}\frac{1}{\sqrt{2}^6} + \frac{3}{2}\frac{1}{\sqrt{2}^2} - 3\frac{1}{\sqrt{2}^4} = \frac{3}{2}\cdot\left(\frac{1}{8} + \frac{1}{2} - \frac{1}{2}\right) = \frac{3}{16} \neq 0$

no solutions

$\left.\begin{aligned} y(0) &= c_1 + c_2 - 3 = 0 \\ y'(0) &= -6c_1 - 2c_2 + 12 = -2 \end{aligned}\right\} \begin{aligned} c_1 &= 2 \\ c_2 &= 1 \end{aligned}$

$y = 2e^{-6t} + e^{-2t} - 3e^{-4t}$

$y(\ln\sqrt{2}) = \frac{2}{8} + \frac{1}{2} - \frac{3}{4} = 0$

Solution: $f(t) = 2e^{-6t} + e^{-2t} - 3e^{-4t}$

b) $y'' + 2y' + y = 4e^{-x}$

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$

$y_c = c_1 e^{-x} + c_2 x e^{-x}$

$y_p = k x^2 e^{-x}$

$Ly_p = (2 - \cancel{2x} - \cancel{2x} + \cancel{x^2} + \cancel{4x} - \cancel{2x^2} + \cancel{x^2}) k e^{-x} = 4e^{-x} \Rightarrow k = 2$

$(y_p)' = (2xe^{-x} - x^2 e^{-x})k$

$y = c_1 e^{-x} + c_2 x e^{-x} + 2x^2 e^{-x}$

$y(0) = c_1 + c_2 0 + 2 0 = 1 \Rightarrow c_1 = 1$

$y(1) = c_1 e^{-1} + c_2 e^{-1} + 2e^{-1} = e^{-1}$

$c_1 + c_2 + 2 = 1 \Rightarrow c_2 = -2$

$y = (1 - 2x + 2x^2)e^{-x}$

$y(2) = (1 - 4 + 8)e^{-2} = 5e^{-2}$

16Y

(a) State the condition on the partial derivatives of $P$ and $Q$ for the differential form

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$$

to be exact. If this condition is not satisfied, show that the differential form can be made exact by multiplying by an integrating factor of the form $\mu(x)$, provided that

$$\frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

is a function of $x$ only. What is the corresponding condition for the differential form to have an integrating factor of the form $\mu(y)$? [5]

Solve the following differential equations using an integrating factor:

(i)

$$2x\frac{\mathrm{d}y}{\mathrm{d}x} + 3x + y = 0\,,$$

[5]

(ii)

$$(\cos^2 x + y\sin 2x)\frac{\mathrm{d}y}{\mathrm{d}x} + y^2 = 0\,.$$

[5]

(b) Use the change of variables $y(x) = u(x)x$ to solve the differential equation

$$(y - x)\frac{\mathrm{d}y}{\mathrm{d}x} + 2x + 3y = 0\,.$$

[5]

## *Solution(s):*

From user: lester

(a) If $P\,dx + Q\,dy = df$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. If $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, then instead aim to get $\mu(x)P\,dx + \mu(x)Q\,dy = df$. This requires $\frac{\partial}{\partial y}(\mu P) = \frac{\partial}{\partial x}(\mu Q)$ i.e.:

$$\mu\frac{\partial P}{\partial y} = \mu\frac{\partial Q}{\partial x} + Q\frac{d\mu}{dx} \Rightarrow \frac{d\mu}{dx} = \frac{\mu}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right).$$

This will be solvable if the RHS is a function of $x$.

Q.E.D.

(i) $2x\frac{dy}{dx} + 3x + y = 0 \Rightarrow (3x+y)\,dx + 2x\,dy = 0$. This is inexact.

Using previous result: $\frac{1}{\mu}d\mu = \int \frac{1}{2x}(1-2)\,dx \Rightarrow \ln\mu = -\frac{1}{2}\ln x \Rightarrow \mu = \frac{1}{\sqrt{x}}$

With this choice: $\left(3\sqrt{x} + \frac{y}{\sqrt{x}}\right)dx + 2\sqrt{x}\,dy = 0$. This is exact.

By inspection, LHS $= df$ where $f = 2\sqrt{x}y + 2x^{3/2} + c$.

$df = 0 \Rightarrow f = \text{const}$ $\therefore$ soln is $2\sqrt{x}(y+x) = \text{const}$.

(ii) $(\cos^2 x + y\sin 2x)\frac{dy}{dx} + y^2 = 0$. (*) This time $\frac{1}{\mu}d\mu = \int \frac{2y - (2\cos x(-\sin x) + 2y\cos 2x)}{\cos^2 x + y\sin 2x}\,dx$

$$\therefore \ln\mu = \int \frac{\cancel{2y} + 2\cos x\sin x - 2y(\cancel{1} - 2\sin^2 x)}{\cos^2 x + 2y\cos x\sin x}\,dx = \int \frac{2\sin x\,\cancel{(\cos x + 2y\sin x)}}{\cos x\,\cancel{(\cos x + 2y\sin x)}}\,dx = -2\ln\cos x$$

$\therefore \mu = \frac{1}{\cos^2 x} = \sec^2 x$. $\therefore$ (*)$\times(\sec^2 x)$ is exact and reads $(1 + 2y\tan x)\frac{dy}{dx} + y^2\sec^2 x = 0$.

By inspection, this is $\frac{df}{dx}$ where $f = y^2\tan x + y$ $\therefore$ soln is $\underline{y^2\tan x + y = \text{const}}$.

(b) $(y-x)\frac{dy}{dx} + 2x + 3y = 0$. Let $y = vx$, then $x(v-1)\left(v + x\frac{dv}{dx}\right) + 2x + 3vx = 0$

$$\Rightarrow (v-1)\left(v + x\frac{dv}{dx}\right) = -(2+3v) \Rightarrow x\frac{dv}{dx} = \frac{2+3v}{1-v} - v = \frac{2+3v-v+v^2}{1-v}$$

$$\Rightarrow \frac{1}{x}dx = \frac{1-v}{2+2v+v^2}\,dv = \frac{2-(v+1)}{(v+1)^2+1}\,dv \therefore \ln kx = 2\tan^{-1}(v+1) - \frac{1}{2}\ln\left((v+1)^2+1\right)$$

$$\Rightarrow \ln kx = 2\tan^{-1}\left(\frac{y}{x}+1\right) - \frac{1}{2}\ln\left(\left(\frac{y}{x}+1\right)^2+1\right).$$

**17Z**

(a) In the context of matrices, describe what is meant by an eigenvector equation. [3]

Four springs, each having stiffness constant $k$, are used to hold three masses $m$ in a line between two rigid supports. It can be shown that the dynamical behaviour of the system is described by the following three coupled differential equations:

$$\begin{aligned} m\ddot{x}_1 &= -kx_1 + k(x_2 - x_1)\,, \\ m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2)\,, \\ m\ddot{x}_3 &= -k(x_3 - x_2) - kx_3\,, \end{aligned}$$

where $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the positions of the masses, along the line of the springs, relative to their static positions, and the double dot denotes the second derivative with respect to time $t$.

(b) Show that the substitutions

$$x_i = a_i \cos(\omega t)\,, \qquad i = 1, 2, 3,$$

where $a_i$ is independent of $t$, transform the differential equations into a set of simultaneous algebraic equations. [2]

(c) Write down a single matrix equation, in the form of an eigenvector equation, that describes the system. [3]

(d) Find the eigenvalues and normalised eigenvectors that satisfy this equation. [6]

(e) Show that the eigenvectors are orthogonal. [3]

(f) For the solution with the largest value of $|\omega|$, sketch $x_1$, $x_2$ and $x_3$ versus time. [3]

## *Solution(s):*

From user: th541

17Z

## 2015 Paper 2 :

(a) $M\underline{x} = \lambda \underline{x}$ is a matrix/vector equation ($M$ a matrix, $\underline{x}$ a vector, $\lambda$ a scalar). ①

The quantities $\underline{x} \neq 0$ satisfying ① are called eigenvectors of $\underline{x}$ with associated eigenvalues $\lambda$.

(b) $x_i = a_i \cos \omega t \Rightarrow$

$$-m\omega^2 a_1 = -ka_1 + k(a_2 - a_1)$$
$$-m\omega^2 a_2 = -k(a_2 - a_1) + k(a_3 - a_2)$$
$$-m\omega^2 a_3 = -k(a_3 - a_2) - ka_3$$

QED.

(c) In matrix form:

$$M\underline{a} = \lambda \underline{a} \quad \text{where} \quad \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \lambda = \frac{-m\omega^2}{k}, \quad \& \quad M = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

(d) Evecs & Evals of $M$: (Define $\mu = -\lambda$)

Solve:

$$0 = \begin{vmatrix} \mu-2 & 1 & 0 \\ 1 & \mu-2 & 1 \\ 0 & 1 & \mu-2 \end{vmatrix} =$$

$$= (\mu-2)(\mu^2 - 4\mu + 4 - 1) - (\mu - 2 - 0)$$
$$= (\mu-2)(\mu^2 - 4\mu + 2)$$
$$= (\mu-2)(\mu - x_+)(\mu - x_-) \quad \text{where} \quad x_\pm = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \frac{1}{2}\sqrt{8} = 2 \pm \sqrt{2}$$

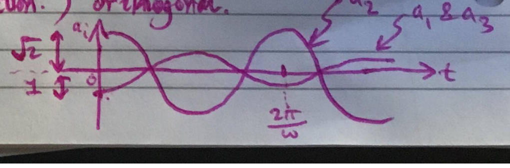
$$\therefore \mu = 2,\ 2+\sqrt{2},\ 2-\sqrt{2} = -\lambda = +\frac{m\omega^2}{k}$$

Evec for $\boxed{\lambda = -2, \mu = 2}$:
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \Rightarrow \underline{a}_{(\lambda=-2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Evec for $\boxed{\lambda = -(2 \pm \sqrt{2})}$:
$$\begin{pmatrix} \pm\sqrt{2} & 1 & 0 \\ 1 & \pm\sqrt{2} & 1 \\ 0 & 1 & \pm\sqrt{2} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \Rightarrow \underline{a}_{(\lambda=-(2\pm\sqrt{2}))} = \frac{1}{2}\begin{pmatrix} -1 \\ \pm\sqrt{2} \\ -1 \end{pmatrix}.$$

(e) These evecs are clearly (by inspection!) orthogonal.

(f) $\text{Biggest } |\omega| = \text{Biggest}\left(\sqrt{\frac{-\lambda k}{m}}\right) = \sqrt{\frac{k}{m}(2+\sqrt{2})}.$




## 18Z

A certain electrical circuit produces a voltage waveform $V_1(t)$ that is periodic in time. The waveform can be written in terms of the Fourier series

$$V_1(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)\,. \qquad (\dagger)$$

(a) If the period of the waveform is $T$, write down an expression for $\omega$. [1]

(b) Starting with ($\dagger$), demonstrate that the coefficients $a_n$ and $b_n$ are given by

$$a_n = \frac{2}{T}\int_0^T V_1(t)\cos(n\omega t)\,\mathrm{d}t\,,$$
$$b_n = \frac{2}{T}\int_0^T V_1(t)\sin(n\omega t)\,\mathrm{d}t\,.$$

[6]

(c) Suppose that the periodic voltage has the form $V_1(t) = V_0\sin(\omega t)$ for $0 \leqslant t \leqslant T/2$ and $V_1(t) = 0$ for $T/2 < t < T$. Derive expressions for the Fourier coefficients, and write out the first 5 non-zero terms (in order of increasing frequency) of the series expansion explicitly. [8]

(d) Create a new function $V_2(t)$ by shifting $V_1(t)$ in time by $T/2$. By noting that $V_1(t) - V_2(t) = V_0\sin(\omega t)$, and without evaluating the Fourier integrals explicitly, write out the first 5 non-zero terms of the series expansion of $V_2(t)$. [5]

### *Solution(s):*

From user: lester

$$V_1(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t) . \qquad (\text{ofo})$$

(a) Period $T$ demands $\omega = \frac{2\pi}{T}$.

(b) Functions of the form $1$, $\cos\left(\frac{n2\pi t}{T}\right)$, $\sin\left(\frac{n2\pi t}{T}\right)$ are orthogonal on $[0,T]$. To find $b_m$, multiply both sides of (ofo) by $\sin(n\omega t)$ and $\int_0^T$. Orthogonality will remove all terms in the integral except that containing the same sine term, ie you will get:

$$\int_0^T V_1(t)\sin(m\omega t)\,dt = \int_0^T b_m \sin^2(m\omega t)\,dt = \frac{T}{2} b_m$$

$$\therefore\ b_m = \frac{2}{T}\int_0^T V_1(t)\sin(m\omega t)\,dt.$$

Same argument works for $a_m$ with $m \geq 1$. ⊛ For $m=0$, we have a slightly different integral since the basis function is "1" so:

$$\int_0^T V_1(t)\,1\,dt = \int_0^T \left(\tfrac{1}{2}a_0\right)1\,dt = \frac{T a_0}{2}$$

which is consistent with the ans $a_m = \frac{2}{T}\int_0^T V_1(t)\cos(m\omega t)\,dt$ from ⊛.

(c) $V_1(t) = \begin{cases} V_0 \sin(\omega t) & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t < T \end{cases}$ $\therefore$

$$a_n = \frac{2}{T}\int_0^{T/2} V_0 \sin(\omega t)\cos(n\omega t)\,dt = \frac{V_0}{T}\int_0^{T/2} \sin((1+n)\omega t) + \sin((1-n)\omega t)\,dt$$

If $n \neq 1$ $\qquad$ $\begin{matrix} 2\omega t = a+b \\ 2n\omega t = a-b \end{matrix}\Big\}$ $\quad$ $\begin{matrix} a = \omega t + n\omega t \\ b = \omega t - n\omega t \end{matrix}$

$$= \frac{V_0}{T}\left[\frac{\cos((1+n)\omega t)}{(1+n)\omega} + \frac{\cos((1-n)\omega t)}{(1-n)\omega}\right]_{T/2}^{0} = \frac{V_0}{\omega T}\left\{\frac{1}{1+n}\left(1-\cos((1+n)\pi)\right) + \frac{1}{1-n}\left(1-\cos((1-n)\pi)\right)\right\}$$

$$= \frac{V_0}{2\pi}\left\{\frac{1}{1+n}\left(1+(-1)^n\right) + \frac{1}{1-n}\left(1+(-1)^n\right)\right\} = \frac{V_0}{2\pi}\left(1+(-1)^n\right)\left(\frac{1-\cancel{n}+1+\cancel{n}}{1-n^2}\right)$$

$$= \begin{cases} \dfrac{2V_0}{\pi(1-n^2)} & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

NB: this answer is consistent with even $n=1$ since $n=1$ is odd, and first term of ⊛ is zero for $n=1$. } So true for all $n$.

$$b_n = \frac{2}{T}\int_0^{T/2} V_0 \sin(\omega t)\sin(n\omega t)\,dt = \frac{2V_0}{T}\int_0^{T/2} \sin\left(\frac{2\pi t}{T}\right)\sin\left(n\frac{2\pi t}{T}\right)dt.$$

But integrand is "even" in $t$ so can double integration range and halve answer.

$$= \frac{2V_0}{2T}\int_{-T/2}^{T/2} \sin\left(\frac{2\pi t}{T}\right)\sin\left(n\frac{2\pi t}{T}\right)dt = \frac{V_0}{T}\begin{cases} T/2 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases} \quad (\text{by orthogonality})$$

$$= \begin{cases} V_0/2 & \text{if } n=1 \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore\ V_1(t) = \frac{V_0}{\pi} + \frac{V_0}{2}\sin(\omega t) + \frac{2V_0}{\pi(1-4)}\cos(2\omega t) + \frac{2V_0}{\pi(1-16)}\cos(4\omega t) + \frac{2V_0}{\pi(1-36)}\cos(6\omega t) + \cdots$$

(d) Given that $V_2(t) = V_1(t) - V_0\sin(\omega t)$, the first five terms of the F.S. of $V_2(t)$ are

$$V_2(t) = \frac{V_0}{\pi} - \frac{V_0}{2}\sin(\omega t) + \frac{2V_0}{\pi(1-4)}\cos(2\omega t) + \frac{2V_0}{\pi(1-16)}\cos(4\omega t) + \frac{2V_0}{\pi(1-36)}\cos(6\omega t) + \cdots$$

**19S***

(a) Explain (without proof) how the method of Lagrange multipliers is used to find the stationary points of the function $f(x, y)$ subject to the constraint $g(x, y) = c$, where $c$ is a constant. How is the method generalized to handle a function of more than two variables subject to one or more constraints? [6]

(b) The function $f(x_1, x_2, x_3, \ldots, x_n)$ of $n$ variables is defined by

$$f = -\sum_{i=1}^{n} x_i \ln x_i \,,$$

where the variables $x_i$ are positive and subject to the constraint

$$\sum_{i=1}^{n} x_i = 1 \,.$$

Show that the stationary point of $f$ subject to this constraint is located where $x_i = 1/n$ for each $i$, and calculate the stationary value of $f$. [7]

(c) If a further constraint

$$\sum_{i=1}^{n} x_i y_i = Y$$

is applied, where $y_1, y_2, y_3, \ldots, y_n$ and $Y$ are given constants, show that the stationary point of the same function $f$ is located instead where

$$x_i = a \, \exp(-b y_i) \,,$$

where $a$ and $b$ are constants. Write down two equations that determine the values of $a$ and $b$. [7]

## *Solution(s):*

From user: lester

(a) Stat points of $f(x,y)$ subject to constraint $g(x,y)=c$ can be found by finding the unconstrained stationary points of $\mathcal{L}(x,y,\lambda) = f(x,y) - \lambda(g(x,y)-c)$ thought of as a function of three variables $x, y$ & $\lambda$. With more variables & more constraints, create an extra $\lambda$ for each constraints. E.g. with vars $\underline{x} = (x, y, \ldots)$ and constraints $g_i(\underline{x}) = c_i$ for $i = 1, 2, \ldots$ then find unconstrained stat. pts of $\mathcal{L}(\underline{x}, \underline{\lambda}) = f(\underline{x}) - \sum_i \lambda_i (g_i(\underline{x}) - c_i)$.

(b) $f = -\sum_{i=1}^{n} x_i \ln x_i$ ; $g(\underline{x}) = 0$ where $g(\underline{x}) = \left(\sum_i x_i\right) - 1$.

Stat pts of $f$ with this constraint solve: $\underline{\nabla} \mathcal{L} = \underline{0}$ where

$$\mathcal{L} = \sum_i \left(-x_i \ln x_i + \lambda\left(x_i - \tfrac{1}{n}\right)\right)$$

$\frac{1}{n}$ since it is inside $\sum_i$ yet must sum to this "1".

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{x_i}{x_i} - \ln x_i + \lambda = 0 \Rightarrow \ln x_i = \lambda - 1$$ (independent of $i$)

meaning that all the $x_i$ are equal to each other.

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_i \left(x_i - \tfrac{1}{n}\right) = 0 \Rightarrow x_i = \tfrac{1}{n}$$ (independent of $i$) since all $x_i$ are identical.

At this soln, the stationary value of $f$ is $f_{min} = -\sum_i \frac{1}{n} \ln\left(\frac{1}{n}\right) = -\ln\left(\frac{1}{n}\right) = \ln n$.

(c) Now add in constraint $h = \left(\sum_i x_i y_i\right) - Y$.

Now $\mathcal{L} = \sum_i \left(-x_i \ln x_i + \lambda\left(x_i - \tfrac{1}{n}\right) + \mu\left(x_i y_i - Y/n\right)\right)$

$\frac{1}{n}$ for same reason as before.

This time,

$$\frac{\partial \mathcal{L}}{\partial x_i} = -\frac{x_i}{x_i} - \ln x_i + \lambda + \mu y_i = 0 \Rightarrow \ln x_i = \lambda - 1 + \mu y_i.$$

$$\Rightarrow x_i = \exp(\lambda - 1 + \mu y_i) = a \exp(-b y_i) \quad \text{where } a = e^{\lambda - 1}$$

and $-b = \mu$. The two constraints that fix $a$ and $b$ are $g$ & $h$:

$$\left.\begin{aligned} g&: \sum_i x_i = 1 \iff 1 = \sum_i a e^{-b y_i} \\ h&: \sum_i x_i y_i = Y \iff Y = \sum_i a y_i e^{-b y_i} \end{aligned}\right\} \text{These fix } a \text{ \& } b.$$

**20T***

A fluid flows with velocity $u(x,t)$ along a channel bounded by walls at $x = 0$ and $x = 1$. The fluid velocity satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2} + G \sin(n\pi x)\,, \qquad (\dagger)$$

where $\nu > 0$ and $G$ are real constants and $n \geqslant 1$ is an integer. It also satisfies the boundary conditions $u(0,t) = u(1,t) = 0$ and the initial condition $u(x,0) = 0$.

(a) For sufficiently large time $t$ the fluid velocity tends to a steady-state solution $u_s(x)$, independent of $t$, that satisfies equation $(\dagger)$ and the same boundary conditions. Find $u_s(x)$. [3]

(b) Now consider

$$\tilde{u}(x,t) = u(x,t) - u_s(x)\,.$$

Find the partial differential equation, analogous to equation $(\dagger)$, satisfied by $\tilde{u}(x,t)$, and show that this equation is independent of $G$. What are the boundary conditions that $\tilde{u}$ must satisfy, and what is the initial condition for $\tilde{u}$? [4]

(c) By means of separation of variables, find a solution for $\tilde{u}$ of the form

$$\tilde{u}(x,t) = f(t)g(x)\,,$$

where $\lim_{t\to\infty} f(t) = 0$. [10]

(d) Hence determine the fluid velocity $u(x,t)$ and the total flow rate

$$Q(t) = \int_0^1 u(x,t)\,\mathrm{d}x\,.$$

[3]

No soution has yet been submitted for this question.