2015 Mathematics (2)

1A

(a) Explain what it means for the differential operator $\mathcal{L}$ to be self-adjoint on the interval $a \leqslant x \leqslant b$. [2]

The eigenfunctions $y_n(x)$ of a self-adjoint operator $\mathcal{L}$ satisfy

$$\mathcal{L}y_n = \lambda_n w y_n \,,$$

for some weight function $w(x) > 0$. Show that for appropriate boundary conditions, eigenfunctions with distinct eigenvalues are orthogonal, i.e.,

$$\int_a^b w(x) y_m^*(x) y_n(x)\, dx = 0$$

for $\lambda_m \neq \lambda_n$. [4]

(b) Consider the eigenvalue problem

$$-\left(1-x^2\right)\frac{d^2y_n}{dx^2} + x\frac{dy_n}{dx} = n^2 y_n \qquad (\star)$$

on the interval $-1 \leqslant x \leqslant 1$, with the boundary conditions $y_n(-1) = 0$ and $y_n(1) = 0$.

(i) Express $(\star)$ in Sturm–Liouville form, and hence determine the weight function $w(x)$. [5]

(ii) By using the substitution $x = \cos\theta$, solve $(\star)$ with the given boundary conditions to show that n must be an integer, and construct the normalised eigenfunctions for $n > 0$. [6]

(iii) Verify explicitly the orthogonality of your eigenfunctions for $n \neq m$. [3]