

2015 Mathematics (2)

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1A

- (a) Explain what it means for the differential operator \mathcal{L} to be self-adjoint on the interval $a \leq x \leq b$. [2]

The eigenfunctions $y_n(x)$ of a self-adjoint operator \mathcal{L} satisfy

$$\mathcal{L}y_n = \lambda_n w y_n,$$

for some weight function $w(x) > 0$. Show that for appropriate boundary conditions, eigenfunctions with distinct eigenvalues are orthogonal, i.e.,

$$\int_a^b w(x) y_m^*(x) y_n(x) dx = 0$$

for $\lambda_m \neq \lambda_n$. [4]

- (b) Consider the eigenvalue problem

$$-(1-x^2) \frac{d^2 y_n}{dx^2} + x \frac{dy_n}{dx} = n^2 y_n \quad (\star)$$

on the interval $-1 \leq x \leq 1$, with the boundary conditions $y_n(-1) = 0$ and $y_n(1) = 0$.

- (i) Express (\star) in Sturm–Liouville form, and hence determine the weight function $w(x)$. [5]
- (ii) By using the substitution $x = \cos \theta$, solve (\star) with the given boundary conditions to show that n must be an integer, and construct the normalised eigenfunctions for $n > 0$. [6]
- (iii) Verify explicitly the orthogonality of your eigenfunctions for $n \neq m$. [3]