2015 Mathematics (2)

This pdf was generated from questions and answers contributed by members of the public to Christopher Lester's tripos/example-sheet solution exchange site http://cgl20.user.srcf.net/. Nothing (other than raven authentication) prevents rubbish being uploaded, so this pdf comes with no warranty as to the correctness of the questions or answers contained. Visit the site, vote, and/or supply your own content if you don't like what you see here.
This pdf had url http://cgl20.user.srcf.net/camcourse/paperpdf/7?withSolutions=1.
This pdf was creted on Fri, 29 Mar 2024 01:18:13 +0000.

1A

(a) Explain what it means for the differential operator $\mathcal{L}$ to be self-adjoint on the interval $a \leqslant x \leqslant b$. [2]

The eigenfunctions $y_n(x)$ of a self-adjoint operator $\mathcal{L}$ satisfy

$$\mathcal{L}y_n = \lambda_n w y_n \,,$$

for some weight function $w(x) > 0$. Show that for appropriate boundary conditions, eigenfunctions with distinct eigenvalues are orthogonal, i.e.,

$$\int_a^b w(x) y_m^*(x) y_n(x)\, dx = 0$$

for $\lambda_m \neq \lambda_n$. [4]

(b) Consider the eigenvalue problem

$$-\left(1-x^2\right)\frac{d^2y_n}{dx^2} + x\frac{dy_n}{dx} = n^2 y_n \qquad (\star)$$

on the interval $-1 \leqslant x \leqslant 1$, with the boundary conditions $y_n(-1) = 0$ and $y_n(1) = 0$.

(i) Express $(\star)$ in Sturm–Liouville form, and hence determine the weight function $w(x)$. [5]

(ii) By using the substitution $x = \cos\theta$, solve $(\star)$ with the given boundary conditions to show that n must be an integer, and construct the normalised eigenfunctions for $n > 0$. [6]

(iii) Verify explicitly the orthogonality of your eigenfunctions for $n \neq m$. [3]

No soution has yet been submitted for this question.