2014 Mathematics (1)

Section A

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Section B

11X

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Solution(s):

From user: ar857

2014 Paper 1

11

a) $\begin{vmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{vmatrix}$ $Tr = 6$

$= -12 + 4\cdot 24 + 7\cdot(-12) = 8\cdot(-12) + 4\cdot 24 = 0$

$\Rightarrow$ at least 1 eigenvalue is $= 0$

b) $\begin{vmatrix} 1-\lambda & -4 & 7 \\ -4 & 4-\lambda & -4 \\ 7 & -4 & 1-\lambda \end{vmatrix} = (1-2\lambda+\lambda^2)(4-\lambda) + (1-\lambda)(-16)$
$+ 4\cdot(-4+4\lambda) + 4\cdot(+4\cdot 7) + 7\cdot(16) - 7\cdot(28-7\lambda)$

$= -\lambda^3 + 2\lambda^2 + 4\lambda^2 - \lambda - 8\lambda + 16\lambda + 16\lambda + 49\lambda + 0$

$= -\lambda\cdot(\lambda^2 - 6\lambda - 72) = -\lambda\cdot(\lambda-12)(\lambda+6)$

$\lambda_1 = 0 \quad \lambda_2 = -6 \quad \lambda_3 = 12$

$\begin{vmatrix} 1 & -4 & 7 \\ 0 & -12 & 24 \\ 0 & 0 & 0 \end{vmatrix}$ for $\lambda_1 = 0$ $\quad e_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\end{pmatrix}$

$\begin{vmatrix} 7 & -4 & 7 \\ -4 & 10 & -4 \\ 7 & -4 & 7 \end{vmatrix}$ for $\lambda_2 = -6$ $\quad e_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$\begin{vmatrix} -11 & -4 & 7 \\ -4 & -8 & -4 \\ 7 & -4 & -11 \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 0 & 18 & 18 \\ 0 & 0 & 0 \end{vmatrix}$ $\quad e_3 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\-1\\1\end{pmatrix}$ for $\lambda_3 = 12$

Verify orthogonality: $\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix} = 0$ ✓
$\begin{pmatrix}1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\1\end{pmatrix} = 0$ ✓
$\begin{pmatrix}1\\0\\-1\end{pmatrix}\begin{pmatrix}1\\2\\1\end{pmatrix} = 0$ ✓

$Ae_1 = 0 \quad Ae_2 = -6e_2 \quad Ae_3 = -12e_2$

c) $Ar\cdot e = 0$

$r = ae_1 + be_2 + ce_3 = (e_1|e_2|e_3)\begin{pmatrix}a\\b\\c\end{pmatrix}$

$Ar = A(e_1\, e_2\, e_3)\begin{pmatrix}a\\b\\c\end{pmatrix}$

$\frac{1}{6}\begin{pmatrix} 1 & -4 & 7 \\ -4 & 4 & -4 \\ 7 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 & -6 & 12 \\ 0 & 0 & -12 \\ 0 & 6 & 12 \end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & -2 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix}$

$Ar\cdot e = 0$

$(1\,0\,1)\cdot(x\,y\,z) = 0 \quad -x+z=0 \quad x=z$

$(1\,-1\,1)(x\,y\,z) = 0 \quad x-y+z=0 \quad y=2x$

yes BUT there is a scaling and inversion too! $\begin{pmatrix}1\\2\\1\end{pmatrix}$

scaling and inversion

$\Rightarrow e = \alpha\begin{pmatrix}1\\2\\1\end{pmatrix}$ $\qquad e = \alpha\begin{pmatrix}1\\2\\1\end{pmatrix}$ d) A projects any vector on a plane $\perp$ to $(1\,2\,1)$

12T

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Solution(s):

From user: ar857

12 2014 Paper I 12

a) $z^3 = (i-1) = \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \sqrt{2}\left(\cos \frac{3}{4}\pi + i\sin \frac{3}{4}\pi\right)$

$z_1 = \sqrt[6]{2}\left(\cos \frac{1}{4}\pi + i\sin \frac{1}{4}\pi\right) = \sqrt[6]{2}\, e^{\frac{1}{4}\pi}$

$z_2 = \sqrt[6]{2}\, e^{(\frac{1}{4}+\frac{2}{3})\pi}$

$z_3 = \sqrt[6]{2}\, e^{(\frac{1}{4}+\frac{4}{3})\pi}$

modulus : $\sqrt[6]{2}$ argumen $\frac{1}{4}\pi + \frac{2\pi n}{3}$ $n = \{0,1,2\}$

b) $\tanh z = -i$ $A = e^z$

$$\frac{A - \frac{1}{A}}{A + \frac{1}{A}} = -i \qquad \frac{\frac{A^2-A}{A}}{\frac{A^2+A}{A}} = -i \qquad \frac{A-1}{A+1} = -i$$

$$\frac{\sinh z}{\cosh z} = -i$$

$\sinh z = -i \cosh z$

$i \sinh z = \cosh z$

$\sin(iz) = \cos(iz)$

$iz = \pi/4 + n\pi$

$z = -i(\pi/4 + n\pi)$

$$\begin{array}{ccccc} 1 & -(4+2i) & 4+5i & -(1+3i) & 2+i \\ 1 & -2-i & -4-4i+1 & 3i+2-1 & \\ 1 & -2-i & i+1 & 0 & \end{array}$$

c) $z = 2+i$

$z^3 - (4+2i)z^2 + (4+5i)z - (1+3i) = 0$

$= (2+i)\left(z^2 - (2+i)z + (i+1)\right) = 0$

$\cancel{4+4i-1 \quad -4(-1+3i+2)}$

$\cancel{4+4i-1+4-12i-8 = \ldots}$

$= (2+i)(z-1)(z-(i+1))$

$z = 1$

$z = (i+1)$

d) $\cos 4\phi = Re\,(e^{\phi i})^4 = Re(\cos\phi + i\sin\phi)^4$

$= Re\,(\cos\phi^4 - 6\cos\phi^2\sin\phi^2 + \sin\phi^4)$

$= \cos\phi^4 - 6(1-\cos\phi^2)\cos\phi^2 + (1-\cos\phi^2)^2$

$= \cos\phi^4 - 6\cos\phi^2 + 6\cos\phi^4 + 1 + \cos\phi^4 - 2\cos\phi^2$

$= 8\cos\phi^4 + 1 - 8\cos\phi^2$

13Y

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Solution(s):

From user: ar857

13 4 2014 I

a) i) $y' + 3y = 8$ $\quad y(0) = 4$

$\frac{dy}{dx} = 8 - 3y$

$\int \frac{1}{8-3y}\,dy = \int dx$

$-\frac{1}{3}\ln(8-3y) = x + c \quad | \cdot 3$

$\ln(1-3y) = -3x + C$

$1 - 3y = e^{-3x} \cdot K$

$y = \frac{-Ke^{-3x} + 8}{3}$ $\qquad y(0) = \frac{-k+8}{3} = 4 \Rightarrow k = -4$

$y = \frac{4}{3}e^{-3x} + \frac{8}{3}$

ii) $y' - y\cos x = \frac{1}{2}\sin 2x$ $\qquad \mu = e^{\int -\cos x} = e^{-\sin x}$

$y' \cdot e^{-\sin x} - \cos x\, y e^{-\sin x} = \sin x \cos x\, e^{-\sin x}$

$\frac{d}{dx}(e^{-\sin x} y) = \sin x \cdot \cos x\, e^{-\sin x}$

$e^{-\sin x} y = -\sin x\, e^{-\sin x} + \int \cos x\, e^{-\sin x} = (-\sin x - 1)e^{-\sin x} + C$

$y = (-\sin x - 1) + C e^{\sin x}$

$y(0) = -1 + c = 0 \Rightarrow c = 1$

$y = e^{\sin x} - (\sin x + 1)$

b) $y'' + 7y' + 12y = 2e^{-3x}$

$\lambda^2 + 7\lambda + 12 = 0 \Rightarrow \lambda_1 = -3 \quad \lambda_2 = -4$

$y_c = c_1 e^{-3x} + c_2 e^{-4x}$

$y_p = K x e^{-3x}$

$Ly_p = ke^{-3x}(-3 - 3 + \cancel{9x} + 7 - \cancel{21x} + \cancel{12x}) = 2e^{-3x} \Rightarrow k = 2$

$y = c_1 e^{-3x} + c_2 e^{-4x} + 2xe^{-3x}$

$\left.\begin{array}{l} y(0) = c_1 + c_2 + 0 = 1 \\ y'(0) = -3c_1 - 4c_2 + 2 - 0 \end{array}\right\} \begin{array}{l} c_2 = -1 \\ c_1 = 2 \end{array}$

$y(x) = 2e^{-3x} - e^{-4x} + 2xe^{-3x}$

plausible

14Z

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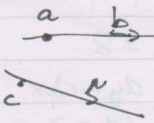
Solution(s):

From user: ar857

(14)

a) $r_1 = a + \lambda\hat{b}$

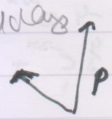
$r_2 = c + \mu\hat{d}$



$d = (a-c)\cdot\frac{b\times d}{|b\times d|}$

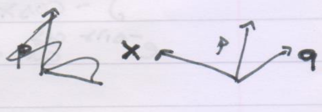
b) $x\cdot p = k$ $\quad x\cdot\frac{p}{|p|} = \frac{k}{|p|}$

$x\cdot\hat{p} = \frac{k}{|p|}$ dist to plane



x is a plane of points, plane is distance k from origin

i) $p\cdot q = 0 \quad p\times q \neq 0$

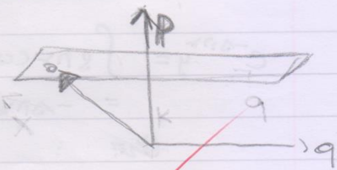


$x = k\frac{p}{|p|^2} + \lambda q + \mu(p\times q)$

point on a plane

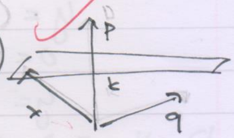
$|p|^2$ I think

vectors in plane



ii) $p\cdot q \neq 0$

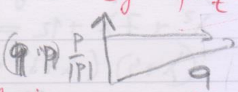
$x = k\frac{p}{|p|^2} + \mu(p\times q) + \lambda\left(q - (p\cdot q)\frac{p}{|p|}\right)$



$x\cdot p = k$

clearly x are posn vectors in plane, but given q: $x = \lambda p + \mu q + \nu p\wedge q$

If $p\cdot q = 0$



$x\cdot p = k \Rightarrow$

① $p\cdot q = 0 \rightarrow \lambda|p|^2 = k \Rightarrow \lambda = \frac{k}{|p|^2}$

② $p\cdot q \neq 0$: $\lambda|p|^2 + \mu p\cdot q = k$

$\therefore \lambda = \frac{k - \mu p\cdot q}{|p|^2}$

$\therefore x = \left(\frac{k-\mu p\cdot q}{|p|^2}\right)p + \mu q + \nu p\wedge q$

c) $x = \frac{\cos t}{\sqrt{1+t^2}} \quad y = \frac{\sin t}{\sqrt{1+t^2}} \quad z = \frac{t}{\sqrt{1+t^2}}$

$r^2 = \frac{\cos^2 + \sin^2 + t^2}{1+t^2} = 1 \quad r = 1$

$\frac{\cos t}{\sqrt{1+t^2}} = r\sin\theta\cos\phi$

$\frac{\sin t}{\sqrt{1+t^2}} = r\sin\theta\sin\phi$

$\frac{t}{\sqrt{1+t^2}} = r\cos\theta$

$\theta = \cos^{-1}\left(\frac{t}{\sqrt{1+t^2}}\right)$

$\phi = t$

$\sin\theta = \frac{1}{\sqrt{1+t^2}}$

$\cos\theta = \frac{t}{\sqrt{1+t^2}}$

$\cot\theta = t$

$\theta = \cot^{-1}(t)$

$r(t) = 1$

$\theta(t) = \cot^{-1}(t)$

$\phi(t) = t$

15W

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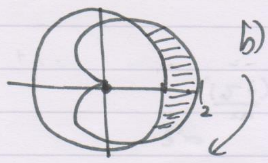
Solution(s):

From user: ar857

15 2014 I 15 (also next page)

a) 

b) $\int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos\phi} r\,dr\,d\phi$

$2\int_0^{\pi/3} -\frac{1}{2}\cos + \frac{1}{2} + \frac{\cos 2\phi}{2}$

$-\frac{\sqrt{3}}{2} + \frac{\pi}{3} + \frac{1}{2}\frac{\sqrt{3}}{2}$

$\cancel{b)}$ $= \int_{-\pi/3}^{\pi/3} \left.\frac{r^2}{2}\right]_{3/2}^{1+\cos\phi} d\phi$

$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 1+\cos\phi^2 + 2\cos\phi - \frac{9}{4} \quad d\phi$

$= \int_0^{\pi/3} 1 + \int_0^{\pi/3} 2\cos\phi + \int_0^{\pi/3} \frac{1}{2} + \frac{\cos 2\phi}{2} + \int_0^{\pi/3} -\frac{9}{4}$

$= \frac{\pi}{3} + 2\cdot\frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{1}{4}\frac{\sqrt{3}}{2} - \frac{\cancel{9}}{\cancel{4}}\pi$

$8 + 4 - 18 \qquad \frac{-6}{24} = -\frac{1}{4}$

$= \pi\cdot\left(\frac{1}{3} + \frac{1}{6} - \frac{3}{4}\right) + \sqrt{3}\cdot\left(1 + \frac{1}{8}\right) = \frac{9}{8}\sqrt{3} - \frac{1}{4}\pi$

b) later, see below

c) $\int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos\phi} \frac{x+y+xy}{x^2+y^2}\,dx\,dy \qquad \frac{x}{x^2+y^2} \quad \frac{r\cos\phi}{r^2}\,r$

$\int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos\phi} \frac{r\,(\cos\phi + \sin\phi + r\sin\phi\cos\phi)}{\cancel{r^2}}\,\cancel{r}\,dr\,d\phi$

$= \int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos\phi} \cos\phi + \sin\phi + r\sin\phi\cos\phi \quad dr\,d\phi$

$\frac{x+y+xy}{x^2+y^2}$ symetric on $y \Rightarrow$

$\int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos\phi} \frac{x}{x^2+y^2}\,dx\,dy = \int_{-\pi/3}^{\pi/3} \int_{3/2}^{1+\cos\phi} \cos\phi\,dr\,d\phi$

$\cancel{\int_{-\pi/3}^{\pi/3} \cos\phi + \cos\phi^2 - \frac{3}{2}\cos\phi} = \int_{-\pi/3}^{\pi/3} \cos + \frac{1}{2} + \frac{\cos 2\phi}{2} - \frac{3}{2}\cos$

$\cancel{= 2\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} - \frac{9}{4}\pi - \frac{3}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8}}$

$= -\frac{\sqrt{3}}{2} + \frac{\pi}{3} + \frac{\sqrt{3}}{4}$

$\boxed{= \frac{\pi}{3} - \frac{\sqrt{3}}{4}}$ $\quad \frac{\pi}{3} - \frac{\sqrt{3}}{4}$

From user: ar857

b)

$$\int_{-\infty}^{\infty} x e^{-\frac{(x-x_0)^2}{a^2}} dx \underbrace{\int_{-\infty}^{\infty} e^{-\frac{(y-y_0)^2}{a^2}} dy \int_{-\infty}^{\infty} e^{-\frac{(z-z_0)^2}{a^2}} dz}_{a^2 \pi}$$

$=$

$$\int_{-\infty}^{\infty} e^{-\frac{(y-y_0)^2}{a^2}} dy \qquad R = \frac{y-y_0}{a} \qquad y = aR + y_0$$

$$dR = \frac{1}{a} dy$$

36

−36

$$= \int_{-\infty}^{\infty} a e^{-R^2} dR = a\sqrt{\pi}$$

48

(−36)

$$\int_{-\infty}^{\infty} a(aR' + \cancel{y}) e^{-R^2}$$

36 −

−48

$$= \int_{-\infty}^{\infty} a^2 R e^{-R^2} + \int_{-\infty}^{\infty} a x_0 e^{-R^2} = a x_0 \sqrt{\pi}$$

$$= a^3 x_0 \pi^{3/2}$$

16Z

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Solution(s):

From user: ar857

2014 I 16

a) $f = 2x^3 + 6xy^2 - 3y^3 - 150x$

$f_x = 6x^2 + 6y^2 - 150$

$f_y = 12xy - 9y^2$

$f_{xx} = 12x \quad f_{yy} = 12x \quad f_{xy} = 12y$

$f_x = 0 \quad x^2 + y^2 - 25 = 0 \quad \cancel{x = \pm 5} \quad x^2 = (5-y)(5+y)$

$f_y = 0 \quad 3y \cdot (4x - 3y) \quad y = 0 \quad x = \frac{3}{4} y \qquad \frac{9}{16} y^2 = 25 - y^2$

Stationary points $\qquad \frac{5}{4} y = \pm 5$

$(5,0)$ min $\quad (3,4)$ saddle $\qquad y = \pm 4$

$(-5,0)$ max $\quad (-3,-4)$ saddle $\qquad x = \pm 3$

b) $g = x^4 + y^4 - 36xy$

$g_x = 4x^3 - 36y = 4 \cdot (x^3 - 9y) \qquad g_{xx} = 12x^2$

$g_y = 4y^3 - 36x = 4 \cdot (y^3 - 9x) \qquad g_{yy} = 12y^2$

$g_{xy} = -36$

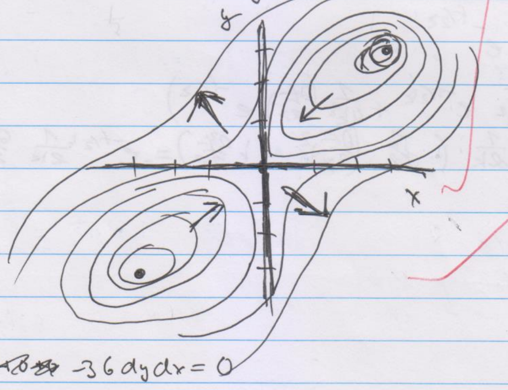
$x^3 = 9y \qquad (0,0)$ saddle

$y^3 = 9x$

$x^9 = 9^3 \cdot 9x \qquad (3,3)$ min

$x^8 = 9^4 \quad x = \pm 3 \quad y = \pm 3 \qquad (-3,-3)$ min

symmetry by $x = y$



~~...~~ $-36\,dy\,dx = 0$

17S

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18S

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19Y*

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20R*

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